Reply to review #2

We thank Referee #2 for the constructive comments that will help us to clarify and improve several points in our manuscript. In the following, we explain how we plan to address the individual issues raised by the reviewer in the revised manuscript.

1. On the KdV approximation

- We fully agree with the reviewer that the assumption that KdV is valid, "cannot be justified on the basis of single-point measurements" (nhess-2022-28-RC2-supplement, p. 1). We would like to point out that we applied vKdV-NLFT as a signal transform, similar to e.g. wavelets or the FFT applied to nonlinear cases. Although we do not know how well the KdV describes the propagation of the measured time series around the measurement site, the KdV does not have to be valid for most of the conclusions of this article, which investigates the results of a signal transform to rogue waves. We do not want to claim that the soliton components in the nonlinear spectrum are physical. We tried to point this out e.g. in the abstract ("Under the hypothesis that the KdV describes the evolution of the sea state around the measurement site well, these results suggest that solitons ...") and the conclusion ("Each measured rogue wave could be associated with at least one soliton in the NLFT spectrum."), but see that this should be pointed out more prominently. We will clarify this in the abstract, the introduction and the conclusion. Our study does not intend to explain the mechanism of rogue wave generation in shallow water. The method should rather be interpreted as a spectral analysis method. We would like to gain insight into the spectral characteristics based on KdV-NLFT at the available measurement site. These spectral characteristics and their differences in samples with and without rogue waves are described in this paper. We would like to point out that in our work, the vKdV-NLFT is applied to a large number of real-world time series for the first time. It is also the first time that certain characteristics of nonlinear spectra could be linked to rogue waves. We thus present a first assessment of the NLFT applied to real measurement data from shallow depths. This is only a first step and future research is needed.
- The reviewer states that "KdV solitons [may usually] be recognized [in time series] by eye" (nhess-2022-28-RC2-supplement, p. 1). We would like to object to this statement and refer to Zabusky and Kruskal (1965), who described the evolution of a sinusoidal-shaped surface elevation, in which solitons eventually form from the background, while not being immediately visible. The observation is reinforced by Brühl & Oumeraci (2016) for the evolution of a long-period cosine wave in very shallow water and in Brühl et al. (2022) for an initially trapezoidal-shaped bore. Here, the solitons that are found by KdV-NLFT, are not immediately visible in the time series, but the surface elevation eventually decomposes into a train of solitons in the far field. While the time series changes with time, the nonlinear spectrum remains invariant. This shows that time series exist, in which KdV solitons are not visible by eye, but may be identified by KdV-NLFT. Another reason for the "invisibility" of the solitons in the time series is that the water surface in the North Sea is not calm before and after the recording of the time series. This means that all existing solitons will continuously interact with the surrounding waves, which makes

them difficult to identify by visual inspection. Figure 1 in Osborne et al. (1991) e.g. demonstrates that solitons do not have to be clearly visible in a real-world measurement.

- The reviewer points out that “the estimated soliton amplitudes are not very large, the solitons do not dominate” (nhess-2022-28-RC2-supplement, p. 1). We have reconstructed a soliton train underlying a time series, by nonlinear superposition of solitons, using the algorithm from Prins & Wahls (2021) (Figure 1 in this document). The time series corresponds to the example in Figure 5 of the preprint. (Note that inverting large soliton spectra is numerically very difficult (Prins & Wahls, 2021). We therefore had to use a shortened time series for the figure.) Figure 1 in this document supports findings by Osborne et al. (1991) that show that the solitons are much lower in amplitude than the maximum waves in the time series. Therefore, we agree with the reviewer, and we have stated so in the conclusion, that in our rogue-wave samples solitons alone cannot be responsible for the formation of the measured rogue waves. The continuous spectrum of the vKdV-NLFT, which actually contains most of the energy in our time series, must account for the remaining parts of the exceptional heights. However, the soliton contribution is not negligible and has the potential to turn a non-rogue wave into a rogue wave. This is concluded from differences in the discrete spectra of samples with and samples without rogue waves.

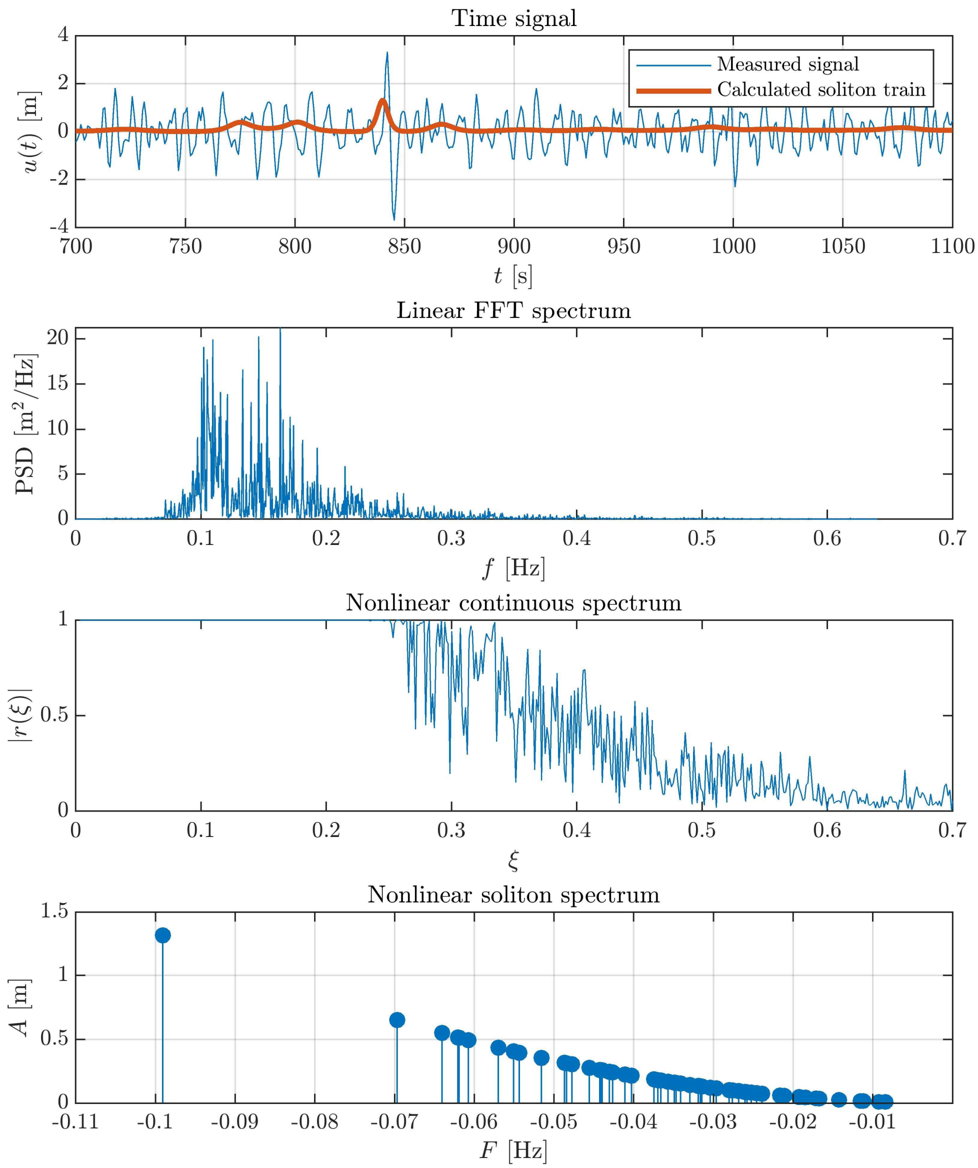

Figure 1: a) time series and soliton train, as calculated from the inverted soliton spectrum. b) linear Fourier spectrum. c) continuous spectrum. d) discrete soliton spectrum.

- We agree with the reviewer to add in the discussion and conclusion that the assumption that waves around our measurement station may be approximated by the KdV equation, cannot be proven based on the available data.
- The reviewer suggests the presence of envelope solitons at varying depths and draws our attention to a study in which “envelope solitons are shown responsible for the wave amplification when the depth increases” (nhess-2022-28-RC2-supplement, p. 1). We agree that varying bathymetry might be an explanation for the enhanced rogue wave occurrence, and will add the corresponding references.
- The reviewer suggests “the oblique interaction of KdV solitons” as an “essentially directional effect” (nhess-2022-28-RC2-supplement, p. 1) as a reason for the enhanced rogue wave occurrence at Norderney. As in the previous comment, we agree with the reviewer that this reason is conceivable, and also, that we cannot assess it in the frames of the KdV equation. All we offer in this article is a signal transformation by NLFT, which

suggests an influence of the presence of shallow-water solitons on rogue wave generation.

- We thank the reviewer for the references on NLS solitons and rogue waves measured in deep water, the accurracy of the NLS equation for strongly nonlinear, the observation of long-lived NLS solitons in the field of strongly nonlinear waves, and KdV soliton content in sinusoids. We will include these in our introduction of the revised article.

2. Comments regarding the text

- l. 44: The reviewer points out that the NLS equation is not limited by $kh = 1.36$. We agree that the formulation in the paper might be misleading and we will clarify this in the revised paper. We will reformulate line 44 in the text to show that the condition is not oriented towards the applicability of the NLS equation, but towards the applicability of the KdV equation.
- l. 83: The reviewer points out that the KdV equation is not the equivalent to the NLS equation, since the former is a wave displacement equation, while the latter is a wave modulation equation. We agree with the reviewer that this formulation is misleading and we will change it accordingly.
- l. 48: We thank the reviewer for the remark that the most unstable perturbations are longitudinal, not oblique. We will replace the word "oblique" by "side-band", as to omit the direction of the perturbation and imply that disturbances arise in the form of side-band modes $\omega(1 \pm \delta)$ to the frequency ω of the initial wave train.
- l. 53: The reviewer points out that "the BFI parameter was introduced for characterization of irregular wave statistics for the first time in [Onorato et al, (2001)]." (nhess-2022-28-RC2-supplement, p. 2) We agree and we will include this information in the text.
- l. 55: The reviewer points out that breather solutions do not explain the physics of the modulational instability, but describe its dynamics. We agree and we will change the text accordingly.
- l. 61: Thank you, we will insert the correct date of the reference.
- l. 67-73: We thank the reviewer for raising the issue of the shallow water definition. In the context of this study, we have termed the range of applicability of the KdV equation as given by Osborne and Petti (1994), p. 1731, and Osborne (1995), p. 2629, shallow water. We acknowledge that, therefore, our definitions of shallow water ($kh < 1.36$ or $h/L < 0.22$) and deep water ($kh > 1.36$ or $h/L > 0.22$) are different from the definitions of shallow ($h/L < 0.05$), intermediate ($0.05 < h/L < 0.5$) and deep water ($h/L > 0.5$) that are used in the engineering context, and that this may lead to confusion. We will clearly define the terminology for 'shallow' that is used in the paper and repeat the *kh* value in line 67. Furthermore, we will change the title to include "shallow depths" rather than "shallow water", as suggested by Referee #1, as to avoid confusion of different shallow water terms.
- l. 78: The reviewer points out that rogue waves in variable depths have been discussed in several studies. We will adjust this text accordingly. We thank the reviewer for the additional reference, which we will include.
- l. 97-99: We will refer to the book of Ablowitz and Segur (1981), where in Chapter 1.7c the asymptotic behaviour is discussed for both the continuous and the discrete spectrum. Since the content in this book is somewhat scattered, we also plan to refer to the paper of Ablowitz and Kodama (1982), who correctly analysed the asymptotic behaviour for the first time.
- l. 102: To solve this issue and avoid misunderstandings, we will include the original citation of Pelinovsky et al. (2000): "the "nonlinear" train should include a soliton".

- l. 103, 257: We agree with the reviewer that the interaction of KdV solitons alone does not lead to the formation of the observed rogue waves. We have stated so in lines 104-105. To make this statement more clear, we will reformulate the words "the interaction between one or multiple solitons with oscillatory waves" as "between one or in principle, multiple solitons, with dispersive waves". In line 257, we presume that the nonlinear interaction of solitons with dispersive waves is the probable cause. For clarification, Figure 1 in this document will be added to the revised version. The figure shows an exemplary time series (blue line in the first plot), its linear FFT spectrum (second plot), the nonlinear continuous spectrum (third plot, which will not be analysed further in this article), and the soliton spectrum (last plot). For visualisation of the role of the solitons in the time series, the first plot shows the soliton train (red line) that is obtained by nonlinear superposition of the solitons (considering their interactions) using the algorithm from Prins and Wahls (2021). This example clearly shows that solitons and their interactions are not solely responsible for the generation of the observed rogue wave.
- l. 117: We thank the reviewer for the additional references, which we will include.
- Figure 2: We agree with the reviewer to add a scale for the distance to the map.
- Equation 1: According to the reviewer, the applied shallow-water threshold is related to the BFI and not to the applicability of KdV. We refer to Osborne and Petti (1994), p. 1731 and Osborne (1995), p. 2629 for the shallow-water threshold in KdV and will add the references to the text.
- l. 155: "1" will be removed from the equation.
- Table 1: The caption will be corrected. We agree with the reviewer that due to the recording of waves at discrete sampling points, there is a possibility that the exact crest of a wave is missed. Therefore, it is conceivable that some of the rogue waves termed "height rogue" are actually underestimated and should belong to the category of "extreme rogues". On the other hand, a possible misinterpretation is conservative, thus, all spectral characteristics attributed to extreme rogue samples are assigned correctly. Furthermore, we think that the impact of such effects is small because the sampling frequency is sufficiently large. For the sufficiency of the sampling frequency of our wave buoy, we would like to cite the sampling theorem (Shannon, 1949): if a continuous time signal contains no frequency components higher than W Hz, it may be completely determined by uniform samples taken at the Nyquist rate of f_s = 2W samples per second. A typical (raw) FFT spectrum of a rogue wave time series from our data is shown in the second plot of Figure 1 in this document. It is seen that its components approach zero at approximately 0.5 Hz and have fully decayed at W = 0.64 Hz. The Nyquist rate f_s = 2W = 1.28 Hz is the measurement frequency of the wave buoy. The signal is therefore oversampled by a factor of more than two. We will add a short discussion of this point in the revised paper.
- l. 211: The reviewer points out that the frequency axis of the discrete soliton spectrum indeed has a physical meaning, which is the inverse duration of a soliton. We agree that the formulation may be misleading: although, theoretically, a soliton has an infinitely long duration, since it does not cross the surface, a mathematical definition of the angular frequency can be established (Equation 10). We refer to the soliton solution of the tKdV (see e.g. Equation (12) in Bruehl et al. (2022)), from which the angular frequency may be obtained. We will therefore reformulate the misleading line 211. The reviewer correctly points out that the amplitude and the frequency of a soliton are related. Hence, the frequency gap in the soliton spectrum may be described either in terms of an amplitude ratio or in terms of a frequency ratio. As stated by the reviewer, the relation between the amplitude and the frequency is quadratic. Consequently, a soliton possessing 80% of the amplitude of the

maximum soliton, has a frequency of $\sqrt{0.8} = 0.89 = 89\%$ of the frequency of the maximum soliton. In other words, a reduction in soliton amplitude by 20% corresponds to a reduction in frequency by only 11%. Since the relative differences in the amplitudes are easier to observe in the amplitude-frequency plots, and the soliton amplitude is a more descriptive parameter than its frequency, furthermore, rogue waves are defined in terms of amplitudes, we have selected to use the amplitude ratio instead of the frequency ratio for the definition of outstanding solitons.

- l. 267, 269: The word “right” corresponds to the statistical term used for a distribution drawn over an x-axis. The reviewer is right that this may be misleading, as the distribution is presented as a box plot, with the vertical axis as a reference. We will therefore rephrase the term “to the right” as “towards higher normalised soliton amplitudes”.
- l. 276/277 and l. 428/429: As already discussed in the beginning, the term "solitons" was mostly used in the manuscript to refer to components in the discrete spectrum of a time series, which might have been confusing. We will explicitly discuss this issue in the abstract, the introduction, and the conclusion.

We would also like to point out that just like the usual Fourier transform is applied also to signals under nonlinear propagation, we apply the vKdV-NFT to signals that may not propagate according to the KdV. Our results show that the vKdV-NFT, when considered purely as a signal processing tool, leads to interesting new characterizations of certain rogue waves, but also demonstrates limits of this approach (at least in the form used here). How far the soliton components in the nonlinear spectrum are physical is an important question, but that question goes beyond what we can answer with our current data. We nevertheless believe that our work is an important step towards bringing the NFT to real-world data, and hope that it motivates future research in that direction.

- Figure 10: We thank the reviewer for this hint and we will add the dimension to the axis.
- Table 3: The relation mentioned by the reviewer is correct for a single soliton, but the soliton spectrum of more complicated time series changes in more complicated ways with the water depth (Figure 2). We prefer not to put this comment to avoid confusion.

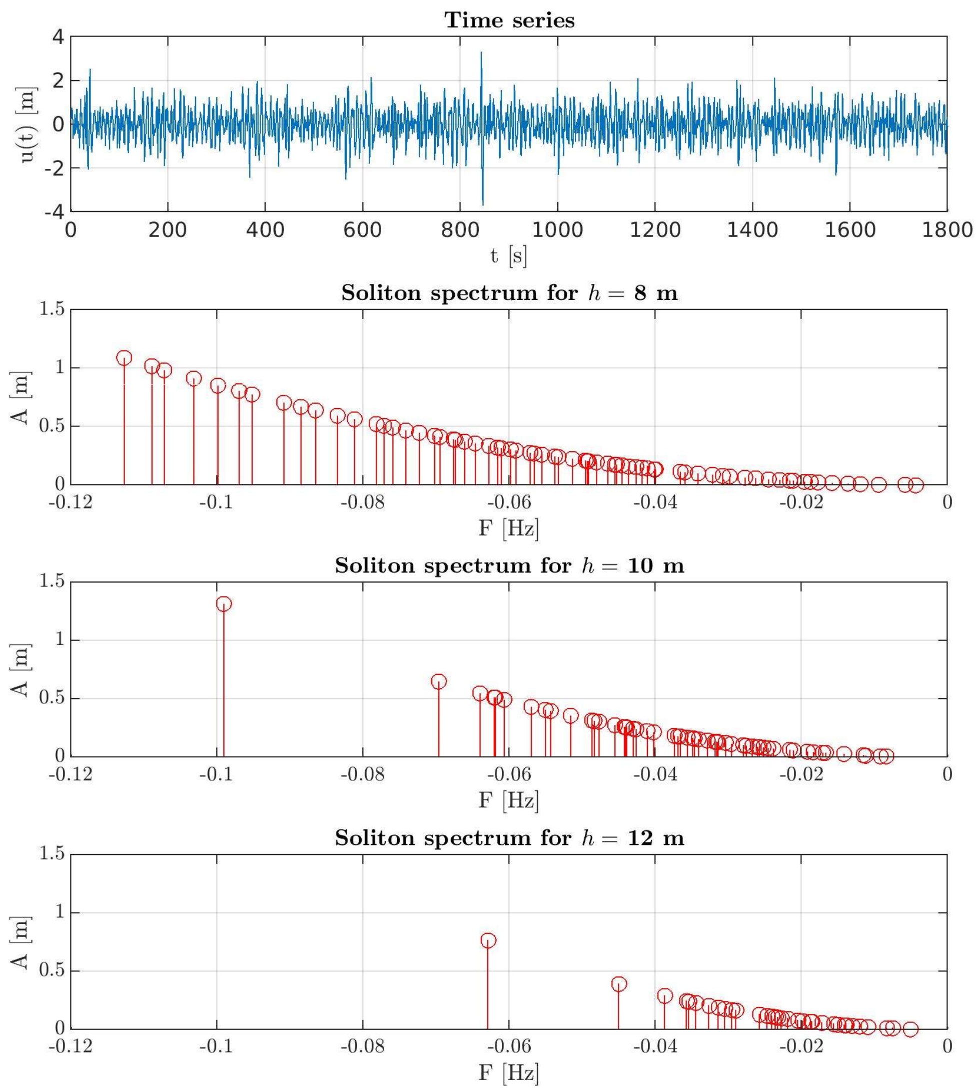

Figure 2: Soliton spectrum of a rogue wave time series for different water depths assumed in the calculation.

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