Flood Routing

The reservoir routing follows continuity equation:

$$\frac{dS}{dt} = Q_l + Q_f - Q_s \tag{1}$$

where S is the storage in the reservoir of the Peñitas Dam, Q_l is the flow generated by the landslide, Q_f, is the flow of tributaries rivers to the site of Peñitas, Q_s is the flow extracted from the Peñitas spillway, and t is the analysis time.

6.2 Storage Capacity Curve

Storage capacity elevation curve for the reservoir may be expressed as:

$$\frac{S-S_o}{S_F-S_o} = \left(\frac{Z-Z_o}{Z_F-Z_o}\right)^{\alpha} \tag{2}$$

where Z is the elevation of the free water surface in the reservoir, So is the storage corresponding to Zo elevation, which will be considered as a conservation level, S_F is storage corresponding Z_F elevation, which can be interpreted as the maximum level that can be reached when Eq. (1) is solved, $\alpha>1$ is a regression constant. The temporal change of water stored:

The time derivation of Eq. (2) yields:

$$\frac{dS}{dt} = \alpha \frac{S_F-S_o}{Z_F-Z_o}\left(\frac{Z-Z_o}{Z_F-Z_o}\right)^{\alpha-1}\frac{dZ}{dt} = \alpha \frac{S_F-S_o}{Z_F-Z_o}\left(\frac{Z-Z_o}{Z_F-Z_o}\right)^{\alpha-1}\frac{dH}{dt} \tag{3}$$

where:

$$H = Z - Z_{cv} \tag{4}$$

is the spillway crest head, and Z_{cv} is crest elevation.

6.3 Hydrograph produced by the landslide

According to Fig. 3, the flow produced by the landslide can be written as

$$Q_l(t) = \begin{cases} 0, t \in (-\infty, 0) \\ Q_{pl}\left(1 - \frac{t}{t_{bl}}\right), t \in (0, t_{bl}) \\ 0, t \in (t_{bl}, \infty) \end{cases} \tag{5}$$

where Q_{pl} is the peak flood and t_{bl} is the base time of the hydrograph. It must be noted that the triangular form of the hydrograph permits an increase in the volume if it is necessary, but can be adopted any form of the hydrograph.

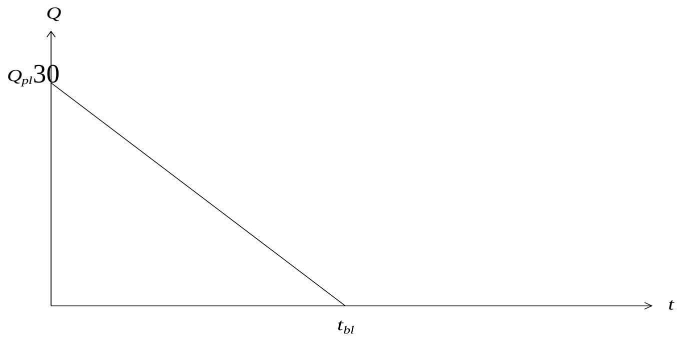

Fig. 3 Discharge law of the hydrograph adopted

6.4 Spillway discharge for the Peñitas Dam

Is usual in hydraulics that discharge follow an exponential law of $H^{\frac{3}{2}}$, and is given by

$$Q_s = \begin{cases} 0, & H < H_o \\ CLH^{\frac{3}{2}}, & H \geq H_o \end{cases} \quad (6)$$

where

$$H_o = Z_o - Z_{cv} \quad (7)$$

C is the discharge coefficient, and L is the spillway length.

Note that if

$$Q_s < Q_l + Q_f, \quad t \in \left(0, t_{pf}\right) \quad (8)$$

then Eq. (6) may be written as

$$Q_s = \begin{cases} 0, & t < 0 \\ CLH^{\frac{3}{2}}, & t \geq 0 \end{cases} \quad (9)$$

In fact,

$$Q_{s,o} \equiv CLH_o^{3/2} \quad (10)$$

is the discharge in the spillway when *t*=0, as is shown in Fig.4.

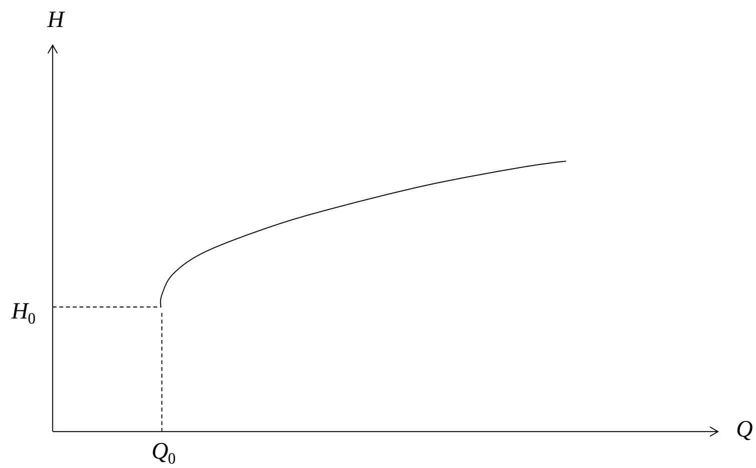

Fig. 4. Discharge Law of the Spillway

6.5 Flood routing reviewed

By substituting Eqs. (3) and (10) in Eq. (1),

$$F_c(H) = \alpha \frac{S_F - S_o}{Z_F - Z_o}\left(\frac{Z - Z_o}{Z_F - Z_o}\right)^{\alpha-1}\frac{dH}{dt} - \left[Q_l(t) + Q_f(t) - CLH^{\frac{3}{2}}\right] = 0, t > 0 \qquad (11)$$

where $Q_l(t)$ y $Q_f(t)$ are given by Eqs. (5) and (6), and $F_c(\cdot)$ is a differential operator that acts over the hydraulic head of the spillway, *H*.

6.6 Flood Routing Discretization

Eq. (13) has no analytical solution for an arbitrary value of α. Thus, a discretization solution based on the trapezoidal rule is done:

$$F_D = (H_j, H_{j+1}; \Delta t_{j+1/2}) \equiv \alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\frac{1}{2}\left(\frac{Z_C + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha-1} + \frac{1}{2}\left(\frac{Z_C + H_{j+1} - Z_o}{Z_F - Z_o}\right)^{\alpha-1}\right]\frac{H_{j+1} - H_j}{\Delta t_{j+\frac{1}{2}}} -$$

$$\left[\frac{Q_{l,j} + Q_{l,j+1}}{2} + \frac{Q_{f,j} + Q_{f,j+1}}{2} - \frac{CL}{2}\left(H_j^{3/2} + H_{j+1}^{3/2}\right)\right] = 0; \; j = 0,1,\ldots. \qquad (12)$$

where

$$H_j \approx H(t_j) \qquad (13)$$

$$H_{j+1} \approx H(t_{j+1}) \qquad (14)$$

Both are discrete approximations of the head values over the spillway crest in time t_j and t_{j+1}. Thus,

$$Q_{l,j} = Q_l(t_j) \qquad (15)$$

$$Q_{l,j+1} = Q_l(t_{j+1}) \quad (16)$$

$$Q_{f,j} = Q_f(t_j) \quad (17)$$

$$Q_{f,j+1} = Q_f(t_{j+1}) \quad (18)$$

In Eq. (14), we can use a time interval variable, defined as

$$\Delta t_{j+1/2} = t_{j+1} - t_j \quad (19)$$

If t_0=0, Eq. (19) stay:

$$t_{j+1} = t_j + \Delta t_{j+\frac{1}{2}} = t_{j-1} + \Delta t_{j-\frac{1}{2}} + \Delta t_{j+\frac{1}{2}} = t_{j-2} + \Delta t_{j-\frac{3}{2}} + \Delta t_{j-\frac{1}{2}} + \Delta t_{j+\frac{1}{2}} = t_o + \sum_{k=0}^{j} \Delta t_{k+1/2} = \sum_{k=0}^{j} \Delta t_{k+1/2} \;, j = 0,1, \ldots. \quad (20)$$

Finally, in Eq. (12), $F_D(\cdot,\cdot;\cdot)$ is a discrete operator that functionally depends on the heads H_j and H_{j+1} and from the parametric point of view, of the interval $\Delta t_{j+1/2}$.

It must also be observed that differences equation (12) is centered in $t_{j+1/2}=(t_j + t_{j+1})/2$, and it can be shown that building a continuum function twice differentiable around $H_j = H(t_j)$ that exactly satisfies Eq. (12), is possible to say:

$$F_D\left(H_j, H_{j+1}; \Delta t_{j+\frac{1}{2}}\right) = 0 \quad (21)$$

Therefore, when differences equation (21) is solved, the differential modified equation $F_C\left(H(t) + O\left(\Delta t^2_{j+\frac{1}{2}}\right)\right) = 0$ is being solved (Warming and Hyett. 1974). It must be noted that the existence of $H(t)$ is guaranteed because the same can be built as a *cubic spline*.

Therefore, also is possible to show that Eq. (12) has a truncated error $T_{j+1/2} = F_D[H(t_j), H(t_{j+1}); \Delta t_{j+1/2}] = O\left(\Delta t^2_{j+\frac{1}{2}}\right)$, (Smith, 1978)

Given that Eq. (12) defines an "ahead march" problem, this equation in finite differences is not lineal in H_{j+1} for known H_j, and then the analytical general solution for arbitrary values of α is not known.

With the objective of giving an analytical solution, a similar strategy to proposed by Beam and Warming (1976) will be used that allows reaching an "implicit factorized scheme."

Remembering the Taylor theorem (Rosenlicht, 1968) for a function twice differentiable, $f = f(x)$ can be written as

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2} f''(\xi)\Delta x^2, \;\; x < \xi < x + \Delta x \,, \quad (22)$$

where the residue has been written in a Lagrangian form.

By identifying x with H_j and $f(x)$ with $\left(\frac{Z_c+H_j-Z_o}{Z_F-Z_o}\right)^{\alpha-1}$, as well as Δx with $H_{j+1}-H_j$, the Taylor theorem (22) can be written as

$$\left(\frac{Z_c+H_{j+1}-Z_o}{Z_F-Z_o}\right)^{\alpha-1} = \left(\frac{Z_c+H_j-Z_o}{Z_F-Z_o}\right)^{\alpha-1} +$$

$$(\alpha-1)\frac{(Z_c+H_j-Z_o)^{\alpha-2}}{(Z_F-Z_o)^{\alpha-1}}\left(H_{j+1}-H_j\right)+\frac{(\alpha-1)(\alpha-2)}{2}\frac{\left(Z_c+H_{j+\beta}-Z_o\right)^{\alpha-3}}{(Z_F-Z_o)^{\alpha-1}}\left(H_{j+1}-H_j\right)^2; \quad 0<\beta<1 \qquad (23)$$

Now identifying x with H_j, $f(x)$ with $H_j^{3/2}$ and Δx with $H_{j+1}-H_j$ for known H_j, it is possible again to apply Taylor's theorem (22) as

$$H_{j+1}^{3/2} = H_j^{3/2} + \frac{3}{2}H_j^{1/2}\left(H_{j+1}-H_j\right)+\frac{3}{8}H_{j+1}^{-\frac{1}{2}}(H_{j+1}-H_j)^2; 0<\gamma<1 \qquad (24)$$

Obviously

$$H_{j+1}-H_j = O(\Delta t_{j+\frac{1}{2}}) \qquad (25)$$

By substituting Eqs. (23) and (24) in Eq. (22) and considering the definition of differences F_D given in Eq. (12), then:

$$F_D = \left(H_j, H_{j+1}; \Delta t_{j+\frac{1}{2}}\right) \equiv \alpha\frac{S_F-S_o}{Z_F-Z_o}\left[\left(\frac{Z_c+H_j-Z_o}{Z_F-Z_o}\right)^{\alpha-1}\right]\frac{H_{j+1}-H_j}{\Delta t_{j+\frac{1}{2}}} - \left[\frac{Q_{l,j}+Q_{l,j+1}}{2}+\frac{Q_{f,j}+Q_{f,j+1}}{2}-\right.$$

$$\left.\frac{CL}{2}H_j^{\frac{3}{2}}-\frac{3}{4}CLH_j^{\frac{1}{2}}\left(H_{j+1}-H_j\right)\right]+O\left(\Delta t_{j+\frac{1}{2}}^2\right)=0,\ j=0,1,\ldots.. \qquad (26)$$

Thus, without altering the magnitude order of truncated error, i.e. of $O\left(\Delta t_{j+\frac{1}{2}}^2\right)$, from finite differences of truncated given by Eq. (12), it is possible to build the next implicit scheme factorized of second order for the approximate solution of differential equation of flood routing given by Eq. (11), neglecting quadratic terms in $H_{j+1}-H_j$ and obviously in $\Delta t_{j+\frac{1}{2}}$ in Eq. (26):

$$F_D = \left(H_j, H_{j+1}; \Delta t_{j+\frac{1}{2}}\right) \equiv \alpha\frac{S_F-S_o}{Z_F-Z_o}\left[\left(\frac{Z_c+H_j-Z_o}{Z_F-Z_o}\right)^{\alpha-1}\right]\frac{H_{j+1}-H_j}{\Delta t_{j+\frac{1}{2}}}+\frac{3}{4}CLH_j^{\frac{1}{2}}H_{j+1}-$$

$$\frac{1}{2}\left[Q_{l,j}+Q_{l,j+1}+Q_{f,j}+Q_{f,j+1}-\frac{CL}{2}H_j^{\frac{3}{2}})\right]=0, j=0,1\ \ldots. \qquad (27)$$

where

$$H_j \approx H(t_j) \quad (28)$$

$$H_{j+1} \approx H(t_{j+1}) \quad (29)$$

are discrete approximations of head values over the spillway crest that acquires in the times t_j and t_{j+1}. A truncated error can be shown that is given by Eq. (27): $T_{j+1/2} = F_D\left(H(t_j), H(t_{j+1}); \Delta t_{j+\frac{1}{2}}\right) = O\left(\Delta t^2_{j+\frac{1}{2}}\right)$. The approximation order of Eq. (12) is not affected; however, Eq. (26) can be written as

$$\alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] H_{j+1} - \alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] H_j + (\tfrac{3}{4}\Delta t_{j+\frac{1}{2}}) CLH_j^{\frac{1}{2}} H_{j+1} - \frac{1}{2}\Delta t_{j+\frac{1}{2}}\left(Q_{l,j} + Q_{l,j+1} + Q_{f,j} + Q_{f,j+1} - \frac{1}{2} CLH_j^{\frac{3}{2}}\right) = 0; j = 0,1, \ldots \ldots \quad (30)$$

and:

$$H_{j+1} = \frac{\alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] H_j + \frac{1}{2}\Delta t_{j+\frac{1}{2}}\left(Q_{l,j} + Q_{l,j+1} + Q_{f,j} + Q_{f,j+1} - \frac{1}{2} CLH_j^{\frac{3}{2}}\right)}{\alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] + (\frac{3}{4}\Delta t_{j+\frac{1}{2}}) CLH_j^{\frac{1}{2}}} \; j = 0,1, \ldots \quad (31)$$

Recursive Eq. (31) let the calculus of the flood routing over the Peñitas Reservoir and allows the calculation of discharged flows by the spillway that correspond to each interval of time, given by Eq. (31):

$$Q_{s,j+1} \equiv CLH^{\frac{3}{2}}_{j+1}; j = 0,1, \ldots \quad (32)$$

It must be observed that with this analysis, associated to time design flood, must coincide with the flood caused by the landslide, which is unlikely to happen. An analysis with different times in each event is a motive for future research.

Maximum water elevation occurs once the landslide peak flow is reached and is given by equating inflow and outflow discharges as is shown in Fig. 5, ($Q_1 \equiv Q_*$). In other words, the value $H_1 \equiv H_*$ is given by Eq. (31), where the time is given by $t_1 \equiv t_*$, in Eq. (31):

$$Q_* \equiv CLH_*^{\frac{3}{2}} = Q_{pf}\left(1 - \frac{t_* - t_{pf}}{t_{bf} - t_{pf}}\right) \quad (33)$$

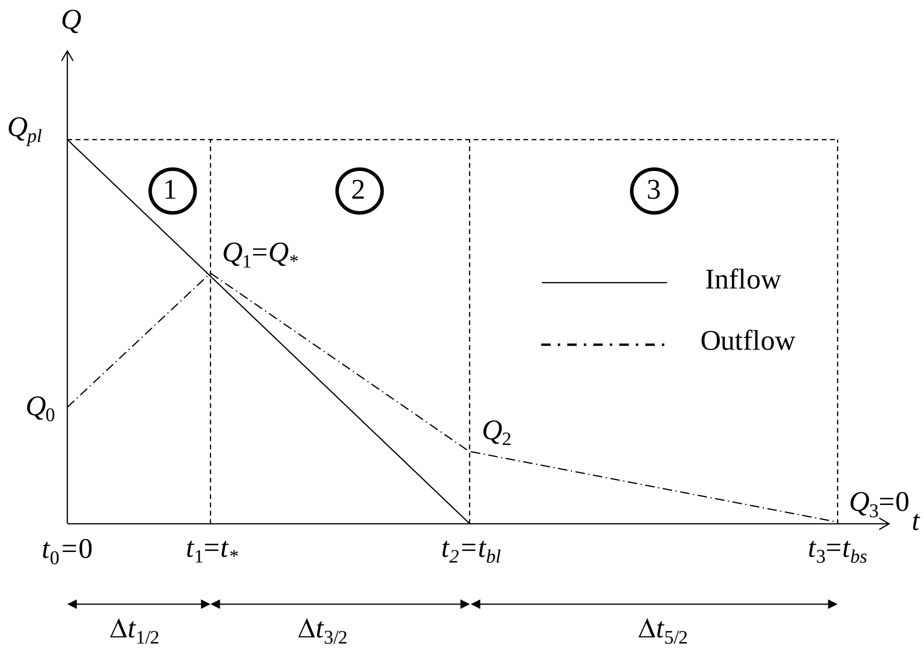

Fig.5 Schematic representation of Inflow-Outflow to Peñitas River.

6.7 Ordinary Risk Case

In the case that only the failure of the natural dam is present without floods from the tributaries, the analysis will be denominated "Ordinary Risk Case," then Eq. (31) continues being applicable with the consideration that $Q_{f,j}=Q_{f,j+1}\equiv 0$, $j=0,1,\ldots$. In this case, Fig. 5 shows that the maximum head belongs to j=0 and is given by:

$$H_{j+1} = \frac{\alpha\frac{S_F-S_O}{Z_F-Z_O}\left[\left(\frac{Z_C+H_0-Z_O}{Z_F-Z_O}\right)^{\alpha-1}\right]H_0 + \frac{1}{2}\Delta t_{\frac{1}{2}}\left(Q_{l,0}+Q_{l,1}-\frac{1}{2}CLH_0^{\frac{3}{2}}\right)}{\alpha\frac{S_F-S_O}{Z_F-Z_O}\left[\left(\frac{Z_C+H_0-Z_O}{Z_F-Z_O}\right)^{\alpha-1}\right]+\left(\frac{3}{4}\Delta t_{\frac{1}{2}}\right)CLH_0^{\frac{1}{2}}} \quad j = 0,1,\ldots \qquad (34)$$

According with this Fig. 5,

$$Q_{l,0} = Q_{p,l} \qquad (35)$$

$$Q_{l,1} = \left(1 - \frac{t_*}{t_{bf}}\right)Q_{p,l} \qquad (36)$$

$$\Delta t_{1/2} = t_* \qquad (37)$$

By substituting Eqs. (35) through (37) in Eq. (34),

$$H_{j+1} = \frac{\alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] H_0 + \frac{1}{2} t_* \left(\left(2 - \frac{t_*}{t_{bl}}\right) Q_{pl} - \frac{1}{2} C L H_0^{\frac{3}{2}}\right)}{\alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_0 - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] + \left(\frac{3}{4} t_*\right) C L H_0^{\frac{1}{2}}} \quad j = 0, 1, \ldots \qquad (38)$$

Analogous to Eq. (32), equating inflow and outflow discharges, when $t = t_*$ (as in Fig. 4)

$$Q_* = C L H_*^{\frac{3}{2}} = \left(1 - \frac{t_*}{t_{bl}}\right) Q_{p,l} \qquad (39)$$

By substituting Eq. (38) in Eq. (39),

$$CL \left\{ \frac{\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] H_0 + \frac{1}{2} t_* \left(2 Q_{pl} - \frac{1}{2} C L H_0^{\frac{3}{2}}\right) - Q_{pl} \frac{t_*^2}{2 t_{bl}}}{\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_0 - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] + \left(\frac{3}{4} t_*\right) C L H_0^{\frac{1}{2}}} \right\}^{3/2} = \left(1 - \frac{t_*}{t_{bl}}\right) Q_{p,l}$$

(40)

Equation (40) is not linear in t_* and can be expressed as a polynomial equation of sixth degree. By the Abel impossibility theorem, it is not possible obtain an explicit solution; therefore, an alternative method is proposed as the one used before for determining t_*. Let now

$$A = \alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] H_0 \qquad (41)$$

$$B = \frac{1}{2}\left(2 Q_{pl} - \frac{1}{2} C L H_0^{\frac{3}{2}}\right) \qquad (42)$$

$$D = \alpha \frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] \qquad (43)$$

$$E = \frac{3}{4} C L H_0^{\frac{1}{2}} \qquad (44)$$

By expanding the left member of Eq. (40) in Taylor series, we have (as in Eqs. (38) and (39) through (44)):

$$\left\{ \frac{\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] H_0 + \frac{1}{2} t_* \left(2 Q_{pl} - \frac{1}{2} C L H_0^{\frac{3}{2}}\right) - Q_{pl} \frac{t_*^2}{2 t_{bl}}}{\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_0 - Z_o}{Z_F - Z_o}\right)^{\alpha - 1}\right] + \left(\frac{3}{4} t_*\right) C L H_0^{\frac{1}{2}}} \right\}^{\frac{3}{2}} = \left(\frac{A + B t_* + B' t_*^2}{D + E t_*}\right)^{3/2} = \left(\frac{A}{D}\right)^{3/2} +$$

$$\frac{3}{2}\left(\frac{A}{D}\right)^{1/2} \frac{BD - AE}{D^2} t_* + O(\Delta t_{\frac{1}{2}}^2) \qquad (45)$$

By neglecting the terms of $O(\Delta t_{\frac{1}{2}}^2)$ in this equation, by substituting the result in Eq. (39) and by solving for t_*, we have

$$t_* = \frac{Q_{pl} - CL\left(\frac{A}{D}\right)^{3/2}}{\frac{3}{2}CL\left(\frac{A}{D}\right)^{1/2}\left(\frac{B}{D} - \frac{AE}{D^2}\right) + \frac{Q_{pl}}{t_{bl}}} \tag{46}$$

From Eqs. (41) through (44), we have

$$\frac{A}{D} = H_0 \tag{47}$$

$$\frac{B}{D} = \frac{Q_{pl} - \frac{1}{4}CL\left(\frac{A}{D}\right)^{3/2}}{\frac{S_F - S_O}{Z_F - Z_O}\left[\left(\frac{Z_C + H_0 - Z_O}{Z_F - Z_O}\right)^{\alpha-1}\right]} \tag{48}$$

$$\frac{E}{D} = \frac{3}{4}\frac{CL(H_0)^{1/2}}{\alpha\frac{S_F - S_O}{Z_F - Z_O}\left[\left(\frac{Z_C + H_0 - Z_O}{Z_F - Z_O}\right)^{\alpha-1}\right]} \tag{49}$$

Hence,

$$\frac{B}{D} - \frac{AE}{D^2} = \frac{Q_{pl} - CLH_0{}^{3/2}}{\alpha\frac{S_F - S_O}{Z_F - Z_O}\left[\left(\frac{Z_C + H_0 - Z_O}{Z_F - Z_O}\right)^{\alpha-1}\right]} \tag{50}$$

By substituting Eqs. (47) through (50) in Eq. (45),

$$t_* = \frac{Q_{pl} - CLH_0^{3/2}}{\frac{3}{2}CLH_0{}^{1/2}\frac{Q_{pl} - CLH_0{}^{3/2}}{\alpha\frac{S_F - S_O}{Z_F - Z_O}\left[\left(\frac{Z_C + H_0 - Z_O}{Z_F - Z_O}\right)^{\alpha-1}\right]} + \frac{Q_{pl}}{t_{bl}}} \tag{51}$$

By finally substituting Eq. (51) in Eq. (38), the explicit expression for the maximum head is obtained:

$$H_* = \frac{\left\{\alpha\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_j - Z_o}{Z_F - Z_o}\right)^{\alpha-1}\right]H_0 + \frac{1}{2}\left[\frac{Q_{pl} - CLH_0^{\frac{3}{2}}}{\frac{3}{2}CLH_0{}^{\frac{1}{2}}\left[\frac{Q_{pl} - CLH_0{}^{\frac{3}{2}}}{\alpha\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_0 - Z_o}{Z_F - Z_o}\right)^{\alpha-1}\right]} + \frac{Q_{pl}}{t_{bl}}\right]}\right]\left[2Q_{pl} - \frac{Q_{pl}}{t_{bl}}\frac{\left[\left(Q_{pl} - CLH_0^{\frac{3}{2}}\right) - \frac{1}{2}CLH_0^{\frac{3}{2}}\right]}{\left[\frac{Q_{pl} - CLH_0{}^{3/2}}{\alpha\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_0 - Z_o}{Z_F - Z_o}\right)^{\alpha-1}\right]} + \frac{Q_{pl}}{t_{bl}}\right]}\right]\right\}}{\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_0 - Z_o}{Z_F - Z_o}\right)^{\alpha-1}\right] + \left(\frac{3}{4}\right)CLH_0^{\frac{1}{2}}\frac{Q_{pl} - CLH_0^{\frac{3}{2}}}{\frac{3}{2}CLH_0{}^{\frac{1}{2}}\frac{Q_{pl} - CLH_0{}^{\frac{3}{2}}}{\alpha\frac{S_F - S_o}{Z_F - Z_o}\left[\left(\frac{Z_c + H_0 - Z_o}{Z_F - Z_o}\right)^{\alpha-1}\right]} + \frac{Q_{pl}}{t_{bl}}}}$$

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