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Natural Hazards and Earth System Sciences
Discussions

# Brief communication: The occurrence of rogue waves in the interior of the oceans: A modelling and computational study

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**Abstract.** The occurrence of unexpectedly large displacements in the interior of the oceans is studied through the dynamics of packets of internal waves, where the evolution is governed by the nonlinear Schrödinger equation. The case of constant buoyancy frequency permits analytical treatment. While modulation instability for surface waves only arises for sufficiently deep water, rogue internal waves may occur if the fluid depth is shallow. The dependence on the stratification parameter and choice of internal modes can be demonstrated explicitly. The spontaneous generation of rogue waves is tested by numerical simulations.

## 1 Introduction

Rogue waves are unexpectedly large displacements from equilibrium positions or otherwise tranquil configurations. Oceanic rogue waves obviously pose immense risk to marine vessels and offshore structures (Dysthe, et al., 2008). After these waves were observed in optical waveguides, studies of such extreme and rare events have been actively pursued in many fields of science and engineering (Onorato et al., 2013). Within the realm of oceanic hydrodynamics, the occurrence of rogue waves in coastal regions has been recorded (Nikolkina and Didenkulova, 2011; O'Brien et al., 2018).

Theoretically the propagation of weakly nonlinear, weakly dispersive narrow-band wave packets is governed by the nonlinear Schrödinger equation, where the dynamics is dictated by the competing effects of second order dispersion and cubic nonlinearity (Ablowitz and Segur, 1979). Modulation instability of plane waves and rogue waves can then occur only if dispersion and cubic nonlinearity are of the same sign. For surface wave packets on a fluid of finite depth, rogue modes can emerge for $kh > 1.363$ where $k$ is the wavenumber of the carrier wave packet and $h$ is the water depth. Hence conventional understanding is that such rogue waves can only occur if the water depth is sufficiently large.

Other fluid physics phenomena have also been considered, such as the effects of rotation (Whitfield and Johnson, 2015) or the presence of shear current (Liao et al., 2017). While such considerations may change the numerical value of the threshold (1.363), the requirement of water of large depth is probably unaffected. For wave packets of large wavelengths, dynamical models associated with the shallow water regime have been employed (Didenkulova and Pelinovsky, 2016), such as the well-known Korteweg-de Vries and Kadomtsev-Petviashvili types of equations (Grimshaw et al., 2010, 2015), which may also lead to modulation instability under several special circumstances.

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The goal here is to establish another class of rogue wave occurrence through the effects of density stratification, namely, internal waves in the interior of the oceans. The asymptotic multiple scale expansions for internal wave packets under the Boussinesq approximation also yield the nonlinear Schrödinger equation (Grimshaw, 1977, 1981; Liu and Benney, 1981). When the buoyancy frequency is constant, modulation instability in one horizontal space dimension will only occur for $kh < k_c h = 0.766n\pi$ where the fluid is confined between rigid walls distance $h$ apart, $n$ is the vertical mode number of the internal wave, and the critical wave number $k_c$ given by:

$$k_c = \frac{n\pi}{h}\left(4^{1/3} - 1\right)^{1/2} . \tag{1}$$

The important point is not just a difference in the numerical value of the cutoff, but rogue waves now occur for water depth ***less*** than a certain threshold. Our contribution is to extend this result. The likelihood of occurrence of internal rogue waves is: (i) determined by estimation of the growth rate of modulation instability, and (ii) elucidated by a numerical simulation of emergence of rogue modes with the optimal modulation instability growth rate as the initial condition.

## 2 Formulation

### 2.1 Nonlinear Schrödinger theory for stratified shear flows

The dynamics of small amplitude (linear) waves in a stratified shear flow with the Boussinesq approximation is governed by the Taylor-Goldstein equation ($\phi(y)$ = vertical structure, $k$ = wavenumber, $c$ = phase speed, $U(y)$ = shear current):

$$\phi_{yy} - \left(k^2 + \frac{U_{yy}}{U - c}\right)\phi + \frac{N^2\phi}{(U - c)^2} = 0 , \tag{2}$$

where $N$ is the Brunt-Väisälä frequency or more simply 'buoyancy frequency' ($\bar{\rho}$ is the background density profile):

$$N^2 = -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dy} . \tag{3}$$

The evolution of weakly nonlinear, weakly dispersive wave packets is described by the nonlinear Schrödinger equation for the complex-valued wave envelope $S$, obtained through a multi-scale asymptotic expansion, which involves calculating the induced mean flow and second harmonic ($\beta$, $\gamma$ being parameters determined from the density and current profiles):

$$iS_\tau - \beta S_{\xi\xi} - \gamma|S|^2 S = 0 , \quad \tau = \varepsilon^2 t , \quad \xi = \varepsilon\left(x - c_g t\right) \tag{4}$$

where $\tau$ is the slow time scale, $\xi$ is the group velocity ($c_g$) coordinate and $\varepsilon$ is a small amplitude parameter.

### 2.2 Constant buoyancy frequency

For the simple case of **constant** buoyancy frequency $N_0$, the formulations simplify considerably in the absence of shear flow ($U(y) = 0$). The linear theory Eq. (2) yields simple solutions for the mode number $n$:

$$N = N_0 , \quad \phi = \sin\left(\frac{n\pi y}{h}\right) , \tag{5}$$

with the dispersion relation, phase velocity ($c$) and group velocity ($c_g$) given by

$$\omega^2 = \frac{k^2 N_0^2}{\frac{n^2\pi^2}{h^2} + k^2} , \qquad c = \frac{\omega}{k} , \qquad c_g = \frac{d\omega}{dk} , \qquad c_g = \frac{c}{1 + \frac{k^2 h^2}{n^2\pi^2}} . \tag{6}$$

The subsequent nonlinear analysis yields the coefficients of the nonlinear Schrödinger equation in explicit forms:

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$$\beta = \frac{3n^2\pi^2c^2}{2h^2kN_0^2}\left(c - c_g\right), \qquad \gamma = -\frac{6N_0^2kc_g^3\left(c - c_g\right)}{c^4\left(c^3 - 4c_g^3\right)}. \tag{7}$$

A plane wave solution for Eq. (4) (or physically a continuous wave background of amplitude $A_0$) is

$S = A_0 \exp[-i\gamma A_0^2\tau]$ .

Small disturbances with modal dependence $\exp[i(r\xi - \Omega\tau)]$ will exhibit modulation instability if

(a) $\Omega^2 = \beta r^2(\beta r^2 - 2\gamma A_0^2)$ is negative, i.e. for $\beta\gamma > 0$; calculations using Eqs. (6, 7) lead to $kh < k_c h = 0.766n\pi$ (Eq. (1));

(b) the maximum growth rate is (imaginary part of $\Omega$) $= \Omega_i = |\gamma| A_0^2$ for a special wavenumber given by $\beta^{1/2}r = \gamma^{1/2}A_0$;

(c) the growth rate for long wavelength disturbance is $|\Omega_i/r| = (2\beta\gamma)^{1/2}A_0$ for $r \to 0$.

In terms of significance in oceanography, the constraint $kh < k_c h = 0.766n\pi$ does *not* depend on the constant buoyancy frequency $N_0$. However, it *does* depend on the mode number ($n$) of the internal wave, with the higher order modes permitting a large range of carrier envelope wavenumber and fluid depth for rogue waves to occur.

**3. Computational Simulations**

An intensively debated issue in the studies of rogue waves is the proper condition which may generate or favour the occurrence of such large amplitude disturbances. One suggestion is the role played by long wavelength modes associated with modulation instability, or 'baseband instability' (Baronio et al., 2015). To highlight this effect and to clarify the role of stratification as well as the choice of internal wave modes, numerical simulations are performed where modes with the proper modulation instability growth rate on a plane wave background are selected as the initial condition (Chan and Chow, 2017). Figure 1 shows that rogue waves can occur sooner with greater values of baseband modulation instability. A comprehensive survey of the dependence on the hydrodynamic and oceanic dynamics parameters, $N_0$, $h$, $k$, and $n$, will be conducted in a future study.

**4. Discussions and Conclusions**

An analytically tractable model for packets of internal waves is studied through four input parameters, $h$ (fluid depth), $k$ (wavenumber of the carrier envelope packet), $N_0$ (buoyancy frequency), and $n$ (mode number of the internal wave), with only $h$ and $k$ relevant for surface waves. For internal waves, modulation instabilities and rogue waves now occur for the shallow water regime. With knowledge of baseband instability and supplemented by computer simulations, the likelihood of occurrence of rogue waves is assessed. Remarkably the constant buoyancy frequency may not play a critical role in the condition of occurrence, but the mode number of the internal wave does. For breathers or other pulsating modes, this buoyancy frequency parameter will enter the likelihood estimation and further analytical and computational studies will be valuable (Sergeeva et al., 2014).

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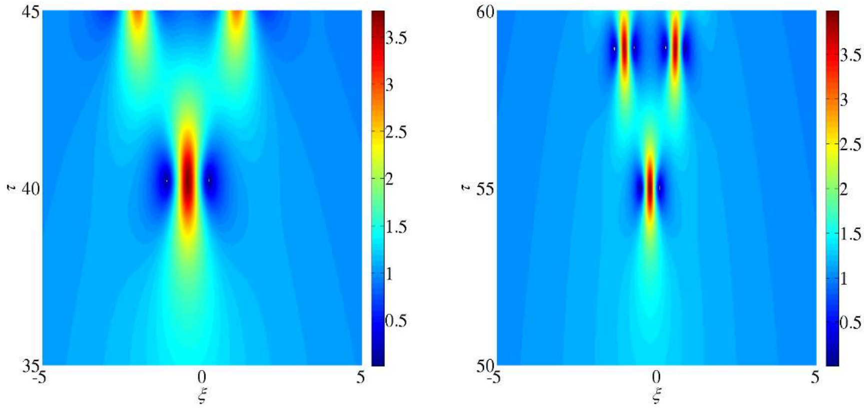


**Figure 1: The emergence of rogue wave modes for different values of modulation instability growth rates with a background continuous wave plus a long wavelength unstable mode as initial condition. Left: $N_0 = 2$, $h = 1.5$, $k = 0.5$, $n = 1$, baseband instability growth rate = 0.0407. Larger gain implies a smaller time is required ($\tau \approx 40$); Right: $N_0 = 2$, $h = 1$, $k = 0.5$, $n = 1$, baseband instability growth rate = 0.0186. Smaller gain implies a longer time is required ($\tau \approx 55$).**

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