THE UNIVERSITY OF HONG KONG

February 28, 2019

The Editorial Board
Natural Hazards and Earth System Sciences

Dear Editors of Natural Hazards and Earth System Sciences,

Thank you for your email of February 21, 2019. We are pleased to submit the revised version of the manuscript, **nhess-2018-238**, by Kwok Wing Chow, Hiu Ning Chan and Roger H. J. Grimshaw. All modifications implemented in response to comments of the referees are highlighted in BLUE and underlined in the revised manuscript. The title has been modified to 'Modulation instability of internal waves in a smoothly stratified shallow fluid with a constant buoyancy frequency', in accordance to a suggestion from a referee. A detailed point-by-point response to the concerns raised by the referees is documented in the following pages. We wish to thank the referees for their valuable opinions and also the Editorial Office for handling the manuscript.

Please feel free to contact us if you need further information. Thank you.

Yours sincerely,

KWC

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nhess-2018-238
Modified title: Modulation instability of internal waves in a smoothly stratified shallow fluid with a constant buoyancy frequency
by Kwok Wing Chow, Hiu Ning Chan and Roger H. J. Grimshaw.

Reply to Referee 3:

We wish to thank Referee 3 for his insightful and supportive comments. We respond in detail as follows.

(1) '*…to replace the title with something like this: "Modulation instability of internal waves in a smoothly stratified shallow water with a constant buoyancy frequency"…*'

Response: This suggestion is adopted, except that the word 'water' is changed to 'fluid' to allow for a more generalized setting. Thank you.

(2) '*The authors present the nonlinear Shrodinger (NLS) equation for internal waves without derivation. This looks fine for the brief communication, however I would emphasise somewhere that it is assumed that the boundary conditions on the water surface is assumed in the fully nonlinear form. Namely this leads to the nonzero nonlinear coefficient in the NLS equation, because the hydrodynamic equations for internal waves with a constant buoyancy frequency are linear*.'

Response: The full derivation of the nonlinear Schrödinger equation governing the evolution of wave packets for a stratified shear flow was given in several earlier works, e.g., Grimshaw 1977, 1981, as well as Liu and Benney, 1981, as cited in the text. The fluid equations for internal waves with a constant buoyancy frequency are linear for plane harmonic waves only. For wave packets there is a wave-induced mean flow, which feeds back to generate a nonlinear term in the asymptotic development. A standard sequence of perturbation calculations and solvability condition will yield the nonlinear Schrödinger equation.

(3) '*…the full version of Taylor–Goldstein equation…suggesting to replace Eq. (2) with the standard equation for internal waves…*'

Response: The form given in the revised manuscript should be the full Taylor-Goldstein equation with shear flow and the buoyancy term.

(4) '…*useful to mention that the authors solve numerically the NLS equation (4)*…'

Response: The following statements were added at the beginning of Section 3:

Modulation instability refers to the growth of small disturbance in a system due to the interplay between dispersive and nonlinear effects (Craik, 1984), and here we examine this by solving the nonlinear Schrödinger equation (Eq. (4)) numerically.

A pseudospectral method with a fourth-order Runge-Kutta scheme for marching forward in time is applied to solve the nonlinear Schrödinger equation (Eq. (4)) numerically.

(5) '…*It is not clear, what are the dimensions of parameters in Figs. 1 and 2? It would be good also to have an estimate for the dimensional maximal growth rate of modulation instability for the first few modes and given buoyancy frequency for h = 500 m*…'

Response: The parameters are dimensionless in Figures 1 and 2 because we are conducting the analysis in a non-dimensional framework. Just like any standard derivation of the nonlinear Schrödinger equation in fluid mechanics, the propagation variable and transverse variable of Eq. (4) are slow time ($\varepsilon^2 t$) and group velocity coordinate. The actual growth pattern of modulation instability thus goes like

Exp[(growth rate) $\varepsilon^2 t$], where t is the time measured in ordinary sense in a laboratory.

Typically a perturbation theory is applied for say $\varepsilon = 0.03$. With a growth rate of order one, the time for the amplitude to amplify by a factor of e (or 2.718 numerically) would be roughly 1,000 seconds, i.e. 16.7 minutes. This is consistent with the time scale of oceanic internal waves. The actual magnitude of velocity can be estimated from $h = 500$ m and the given ε of the perturbation theory. Note that baseband growth rate should be scaled by the wavenumber r of the long wave disturbance. The following statement is added in Section 3: The concrete numerical values of the growth rates in a laboratory frame of reference (time t) can be estimated from definitions used in Eq. (4), i.e. $\tau = \varepsilon^2 t$ and the small amplitude parameter ε actually employed.

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Reply to Referee 2:

We wish to thank Referee 2 for his opinions.

(1) '…*In answering my previous comment (3), the authors state that they are studying "internal rogue waves as opposed to surface waves". Yet, the introduction is strongly focussed on surface rogue waves. This is quite confusing and led to my suggestions for missing references*…'

Response: In contrast to surface rogue waves, there is very little work on this topic of possible rogue waves in the interior of the ocean. The goal of the present work is to propose one such plausible framework. As pointed out in the January 2019 (i.e. previous) revised manuscript, internal solitons may have a large literature but they are ***not*** rogue waves, as a soliton is not localized in time. Nevertheless, we have included some references for internal waves in general in the background discussion (Section 1).

'…*The authors should focus more on the main topic of their work and highlight the new and striking results they have achieved*…'

Response: Our theme of internal rogue waves occurring for the shallow water (long wavelength) regime has been emphasized at several places of the text.

'…*In addition, if one of the goals is to reach a broad audience*,… *beneficial to briefly explain potential relevance of this work in other fileds of physics or engineering*…'

Response: Brief remarks on the importance of internal waves are made and a few references in this field are added (Section 1):

Nearly all experimental and theoretical studies in the literature of rogue waves in fluids focus on surface waves. Our aim here is to investigate a similar scenario for internal waves. Internal waves play critical roles in the transport of heat, momentum and energy in the oceans, and breaking of such waves may have impact on circulation (Pedlosky, 1987). There is a quite substantial literature on observations and theories of large amplitude internal waves in shallow water (Stanton and Ostrovsky, 1998). Many studies concentrate on solitary waves in long wave situations employing the Korteweg-de Vries equation (Holloway et al., 1997), but not on the highly transient modes with a potential of abrupt growth. For relevance in other fields of physics and engineering, the actual derivation of the governing equations may dictate the regime of input parameter values necessary for rogue waves to occur.

(2) (a) '…*not accessible by a broad, general audience. E.g. What are the initial condition? The authors states that they chose a modulation instability mode with the optimised growth rate. What does this mean? What condition should I use if I want to replicate these results?*…'

Response: The topic of modulation instability has been discussed thoroughly in many widely used monographs in fluid mechanics and optics, e.g.,

Craik, A. D. D.: Wave Interactions and Fluid Flows, Cambridge University Press, 1984.

Mei, C. C.: The Applied Dynamics of Ocean Waves, World Scientific, 1989.

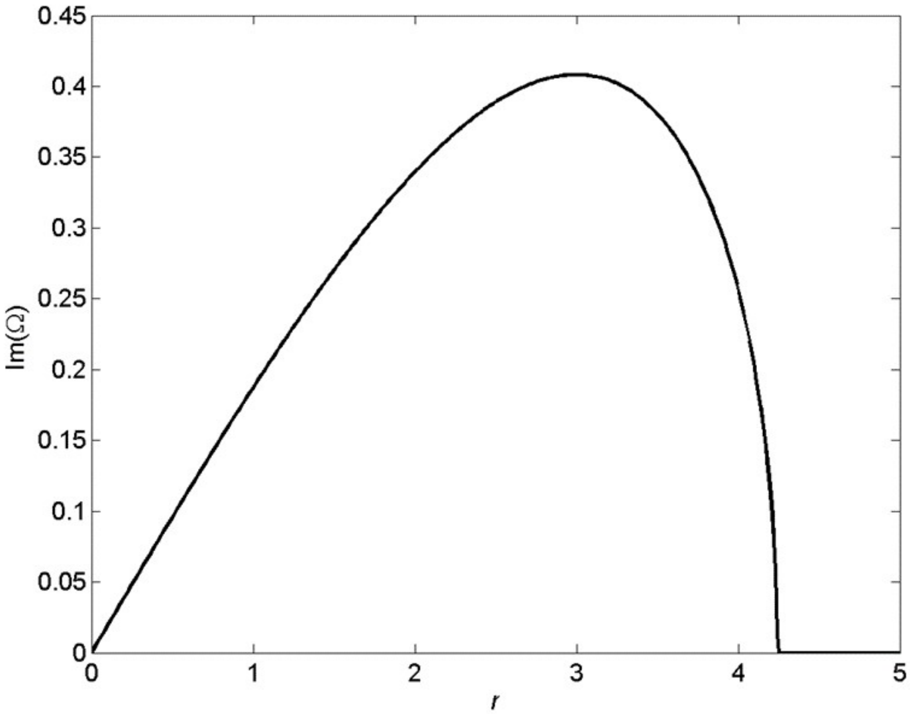
Agrawal, G. P. and Kivshar, Y.: Optical Solitons, Academic Press, 2003.

It may not be fair to the journal to describe all the details of such calculations in this paper. Hence we just add a brief explanation in Section 3 and include the book by Craik:

Modulation instability refers to the growth of small disturbance in a system due to the interplay between dispersive and nonlinear effects (Craik, 1984), and here we examine this by solving the nonlinear Schrödinger equation (Eq. (4)) numerically.

To avoid possible confusion, instead of saying 'proper' or 'optimized' growth rates, we use the words 'maximum' for growth rates and 'baseband modes' in Section 3.

For these types of problems in modulation instability, usually there is a range of unstable wave numbers. The following diagram exhibits the typical growth rate versus the wavenumber:



Our goal is to show that the optimized mode for the generation of rogue wave is a baseband mode at one end of the spectrum with small wavenumber.

(b) '…*The sentence at lines 5-7 at page 4 tries to explain the initial conditions, but it is too technical that general audience would not be able to understand*…'

Response: Numerical simulations with a perturbed plane wave as initial condition have been frequently performed in the literature, starting roughly with say early works on rogue waves about ten years ago (N. Akhmediev, A. Ankiewicz, J. M. Soto-Crespo, *Physical Review E* 80, 026601, 2009). We have also included our own recent work as a reference (Chan, Grimshaw, Chow, *Physical Review Fluids* 2018) in the text, where further details can be found. We choose a special modulation instability mode (one with maximum growth rate) to reduce the time these 'rogue wave like' modes will emerge. The following statement is added in Section 3: This choice of a preferred modulation instability mode as the initial condition is different from other approaches in the literature, such as one using random noise.

(c) '…*Another question was related to the values of the key parameters. Unfortunately, the authors only listed the parameters without providing the requested values, again leaving the reviewer and possible readers unaware of the initial set up of the model*…'

Response: We actually do not understand this request. All the parameter values used in the numerical simulations are given in the figure captions. The corresponding coefficients in the nonlinear Schrödinger equation can readily be computed from Equations (6) and (7). We have added the following sentence in the text (Section 3): Numerical simulations were performed with parameter values appropriate in applications to fluid mechanics.

(3) '…*the manuscript only describes one of condition of density profile with a constant buoyancy frequency* N_0*.* …'

'…*what if N is not constant? This is a relevant question if the goal is to establish the effect of density stratification*…'

Response: The case of constant buoyancy frequency *N* is simplest and allows analytical formulation. The case 'non-constant' *N* will be treated in future research effort, as stated near the end of Section 4: Density profiles with variable buoyancy frequency will also be examined in the future.

'…*How do results change if the value of* N_0 *changes?* …'

Response: Simulations with different N_0 had actually been given in the revised version of January 2019 (Figure 1). In general, changes in parameter values which enhance a higher 'baseband' modulation instability growth rate will cause the rogue wave to appear sooner. It can be observed from Equations (6) and (7) that the baseband growth rate is independent of N_0.

'…*the authors state in the conclusions that constant buoyancy might not play a critical role*...'

Response: We are afraid that those words were taken out of a discussion sequence in an inappropriate manner. The complete description should go like this (Section 4):

'…rogue waves is assessed. Remarkably the constant buoyancy frequency may not play a critical role in the existence condition in terms of focusing, but the mode number of the internal wave does. For breathers or other pulsating modes, this buoyancy frequency parameter will enter the focusing mechanism consideration…'

For rogue waves, N_0 will not enter explicitly in the formulation of focusing as long as it is constant. However, it will appear for breathers and other pulsating modes.

(4) '*…I really don't understand the discussion on kh. For a unidirectional sea state, modulational instability ceases for kh < 1.36, although it can survive if oblique modulations are applied (this has been verified down to kh = 1 I guess)…*'

Response: Firstly, the threshold for instability to arise, i.e. $kh > 1.363$, is valid for surface waves only, and we are trying to investigate internal waves now. Secondly, instability for surface wave occurs for sufficiently deep fluid (i.e. h large). For internal waves, we propose that not only is the numerical limit different, but the qualitative picture differs too, i.e. instability occurs for small h.

'*…The authors state that internal rogue waves appear for kh < 0.766n\pi. The value of modes n ranges from 1 to 5 (in Table 1). Therefore kh is between 2.4 (for n=1) and 12 (for n=5). These values are considerably larger than the 1.36 threshold and definitively are not of shallow water depth conditions and only marginally consider intermediate water depths. This is very confusing; how can the authors claim that internal rogue waves may arise in shallow and intermediate water regimes, for which kh should be lower than say 2? Maybe I am misunderstanding the manuscript,…*'

Response: We are afraid that the referee had indeed misunderstood the formulation here. The integer refers to the mode number of internal waves. For waves in the interior of the fluid, the vertical structures may have zero, one, two…or integer number of nodes (points of zero displacement), and frequently are referred in the literature as the first ($n = 1$), second ($n = 2$), third ($n = 3$)…mode respectively.

'*…Maybe I am misunderstanding the manuscript, but if this is the case, the authors should make a significant effort to explain their work more comprehensively. If the reviewer cannot understand, it is likely that the broad audience would not understand either…*'

We realize the difficulties in explaining the novel concepts of any paper to a general audience. Anyway we attempt to address a broad audience by adding these remarks about introductory ideas on internal waves in this section:

► Mode number n: Internal waves in general display more complex dynamical features than their surface counterparts. As an illustrative example, a given density profile may allow many internal modes characterized by the number of nodes in the vertical structures. This family of allowed states will be generically represented in this paper by an integer n termed mode number.

► Different kh for different n: For a wave packet associated with the first internal mode ($n = 1$), modulation instability or rogue wave can occur for carrier wave number k and shallow fluid of depth h in the range of $kh < 0.766\pi$ or 2.406.

► Reinforce this feature for *n* in Table 1: (mode number of internal wave, with each *n* representing a different vertical structure)

(5) '…*Results presented in section 3 are not particularly new to me, at least in a way they are presented*…'

Response: We cannot change the mathematics of the nonlinear Schrödinger equation (NLSE), but the ways and interpretation of NLSE as applied to say fluid mechanics and optics are dramatically different. Here we are trying to illustrate how the same theory of NLSE, as employed for surface and internal waves, yields different interpretations, e.g. in terms of the buoyancy frequency *N*, depth *h* and internal wave mode number *n*.

'…*striking results that relate to internal wave and deserve a brief communication*…'

Besides the frequently mentioned importance of internal waves in terms of transport as stated earlier, such waves can also play significant roles in underwater acoustics. Hence an abnormally large internally wave appearing abruptly may have physical implications. A remark and references are added in Section 4: Besides their relevance in transport phenomena, internal waves have significant connection with underwater acoustics (Apel et al, 2007; Zhou et al., 1991), and abnormally large internal rogue waves may have physical implications in those aspects.

(6) '…*The authors use an ad-hoc initial condition to achieve maximum instability. Why do the authors claim at line 12 (page 4) that rogue waves emerge spontaneously? It seems to me they are just forcing them to occur and this is far from being spontaneous*…'

Response: We do not agree with the word 'ad-hoc'. The connections between baseband modulation instability and rogue waves from a deterministic approach have been amply demonstrated in the literature (Baronio et al., 2015; Chan et al., 2018 and other works cited there). This choice of preferred mode (baseband mode, Section 3) will allow the rogue wave to emerge sooner than using a random noise as initial condition. One of our goals is to demonstrate the importance of baseband modes in the generation mechanism of internal rogue waves. As the spatial structures of the input and the emerged entities ('rogue wave like' units) differ dramatically, many researchers in this field take this as a demonstration of a 'spontaneous' occurrence.