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Interactive
Comment

Interactive comment on "Numerical modelling of tsunami wave run-up and breaking within a two-dimensional atmosphere–ocean two-layer model" *by* S. P. Kshevetskii and I. S. Vereschagina

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1 The unswer to the referee

First, the author thanks the referee for attention to the the work and for the expressed opinion. We try to answer to the questions.

1–2. The wave breaking is shown in Fig. 5-7. Fig. 5 and 7 shows the splitting of the initial localized wave into two localized waves. Fig. 6 shows the wave breaking with the formation of small-scale waves and the development of mixing within the fluid.

We guess that the referee expects to see the breaking of the surface of the ocean water, when he reads the words about the wave breaking. This is the prevailing stereotype, we often see the breaking of the water surface when waves propagate to a coast.

However, the concept of the wave breaking is much wider, it includes also the wave breaking in thickness of the fluid. The main wave energy is in the interior of the fluid. Therefore, the most interesting processes of the wave breaking are expected also in the bulk fluid.

At the moment, the tsunami wave propagation is modeled perfectly with using the shallow water equations. The shallow water equations describe the formation of wave structure similar to the shock wave, and we see, though inaccurately, the occurrence of breaking of the water surface. However, the shallow water equations are based on averaging of the hydrodynamic equations along the vertical; so these equations do not describe breaking of waves within the fluid. Thus, the processes of breaking of waves within fluid almost fall out of the theoretical analysis.

However, we all know that when a wave approaches a shore, the extremely strong turbulence is formed, which indicates very developed processes of breaking of waves within the fluid. Our model closes this gap and it aims to explore this process. Of course, we cannot simulate the developed turbulence because the turbulence is one of the difficult problems of hydrodynamics. However, the simulation of the first phase of the process is quite possible. Fig. 6 in the paper demonstrates the processes of breaking of the wave inside the fluid. Fig. 6 shows only a rough structure, but in the beginning of the research is important to get a general idea about the process. We found that the development of this process depends on the shape of the bottom and pre-examined this relationship.

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3. The equations (1) of the paper are the two-dimensional fundamental hydrodynamic equations for incompressible fluid in the gravity field, without any simplifications. Only viscosity is not taken into account. All existing two-dimensional models of tsunami waves, without an exception, are derived from the system of equations (1).

4. The equation for density

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0 \tag{1}$$

is fundamental and is fair for any fluid, including the case of an incompressible fluid. We use equation (1) because it has the form of a conservation law. Equations in the form of conservation laws are necessary for the mathematical apparatus of generalized (weak) solutions used by the authors and for developing of conservative numerical methods. See also note 8.

5. The equation for Ψ is the fundamental equation of movement of an incompressible fluid, in case of two space dimensions. The derivation is easy, and we show it below. We consider Euler's equations for fluid, placed in the gravity field. The equations are taken in the form of fundamental conservation laws

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho w}{\partial z} = -\frac{\partial p}{\partial x} \tag{2}$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho u w}{\partial x} + \frac{\partial \rho w^2}{\partial z} = -\frac{\partial p}{\partial x} - \rho g \tag{3}$$

We differentiate Eq. (2) with respect to z and Eq. (2) with respect to x. Then we subtract the second result from the first one, and we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho w}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \rho w}{\partial t} + \frac{\partial \rho u w}{\partial x} + \frac{\partial \rho w^2}{\partial z}\right) = \frac{\partial}{\partial x}\rho g \tag{4}$$

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Now we substitute $u = \frac{\partial\Psi}{\partial z}$, $w = -\frac{\partial\Psi}{\partial x}$ into Eq. (4), and we immediately come to the equation for Ψ of the paper. You see that (1) is equivalent to the Euler's equations for an incompressible fluid.

6. Dear referee, we can prove mathematically that the current lines are closed. We guess, when you have seen the closed streamlines in the figures, you have decided what a circular mass transfer happens and hence the rotational motion of the fluid takes place. It is a hasty, erroneous conclusion. We are dealing with a wave process. The stream function is time-dependent at each spatial point. Let us consider an arbitrary point. If at some instant the stream function shows that the fluid moves upward at this point, at another instant the stream function will be different and the fluid motion will be different, including the particle can move downward. Thus, the motion of fluid particles is of oscillatory character, in spite of the fact that the current lines are closed.

7. We solve a two-dimensional problem. The horizontal scale of a wave far from the coast is much larger than the vertical one; and the horizontal fluid velocity is much more than the vertical fluid velocity. It means that the fluid movement in the wave far from the coast is quasi-one-dimensional. It means also that the fluid movement is close to a potential one. This fact is shown in the listed pictures: the current lines are almost horizontal far from the coast.

8. We use the mathematical apparatus of weak/generalized solutions. Within this approach, no interboundary conditions at the air-ocean interface is required.

   The theory of weak/generalized solutions is given, for example, in chapter 17 of

   R.Richtmayer . "Principles of advanced mathematical physics", v1. 1978. Springer-Verlag

   The formulas (17.5.8) and (17.5.9) of the book together with the paragraph below give an answer to the question of dear referee.

In shorts:

Petter Lax in 1954-1955 has suggested to reformulate the hydrodynamic equations in an integral form: in the form of fundamental conservation laws of mass, momentum and energy. The integrands do not demand differentiability; even continuity of the integrands is not required. Therefore, the integral formulation of the hydrodynamic equations allow to work with non-differentiable solutions, and even with the solutions having ruptures, including density ruptures.

The ocean-atmosphere interface is a rupture of a contact type. In Richtmayer's book, it is shown that the following outcomes follow the definition of a weak solution: the pressure changes continuously when the point crosses such a rupture, and the velocity component which is perpendicular to the rupture surface, is continuous. Thus, the both, the pressure continuity on the ocean-atmosphere interface and the condition of continuity of normal velocity to the ocean-atmosphere interface automatically follow the definition of a weak solution of hydrodynamic equations.

9. The soliton models of a tsunami waves are well known; these are examples of solitary waves. The soliton model of surface gravity waves based on the Korteweg de Vrice equation is well known as well; it is an example of a solitary gravity wave. Therefore, the authors found it possible to omit the description of these terms in the hope that they are well known. The initial conditions to our problem is prepared like the initial condition for the soliton model of tsunami waves. We did not make justification of the initial conditions in details, because we only wanted to show that the problem can be solved in the full statement without simplifications of equations. Comparisons of simulations with soliton models of tsunami waves and comparisons of simulations with experimental observations would be useful and interesting certainly, but we hope that we will do it in future.

10. It is incorrect, that the numerical method is not published. The used numerical

method is published in

Kshevetskii S.P. Study of Vortex Breakdown in a stratified Fluid. Computational Mathematics and Mathematical Physics, 2006. Vol. 46. N11. pp.1988-2005.

We refer to this paper in our article, and the paper is easily accessible

http://link.springer.com/article/10.1134%2FS0965542506110133

The numerical methods for the shallow water equations are rather simple, and the authors easily write the main formulas of numerical methods in each paper. The numerical methods for the full system of hydrodynamic equations for an incompressible fluid often are exclusively difficult and bulky. Therefore, the numerical methods for the full hydrodynamic system usually are published only once, in the journal specializing on numerical methods. Then the authors refer to the publication. The author has made the same. Nevertheless, because of the referee wishes, the author has described the numerical method in the Appendix. It is a short extract from the cited paper.

11. All the figures are given only for an illustration, in order that the reader could understand easily the process of wave propagation within the given model. The presented graphs in the paper are not intended for the control of goodness of the numerical method.

12. Goodness of the numerical model is mainly guaranteed by the proved mathematical theorems (see the author's paper listed). We should take in view also that accuracy of numerical simulation very strongly depends on a computer power. The more powerful computer we run, the smaller numerical scheme steps we can use, and the more precise results we obtain.

13. If the dear referee can give some practical tips in use of English, the author will accept gladly these tips, because the English is not native to the author. Up to the present, the main discussion is concerned only of the mathematical apparatus

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used in the paper. The dear referee surmises that some simplifications have been used, but the author answers that it is not true. The full system of equations has been solved.

2 Appendix

Interactive comment on Nat. Hazards Earth Syst. Sci. Discuss., 2, 3397, 2014.

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4. NUMERICAL SCHEME FOR SOLVING THE EQUATIONS

The equations are approximated using a finite-difference scheme on the grid shown in Fig. 1.

The differentiation operators with respect to coordinates in (2.1) are replaced by second-order accurate finite-difference approximations on the five-point stencil. The discretization over space yields a finite system of ordinary differential equations, which can be solved, for example, by the improved Euler method.

The continuity equation is approximated on the grid shown in Fig. 1 according to the following scheme:

$$\frac{d}{dt}\rho_{i,k} + \frac{\tilde{\rho}_{i+1,k}u_{i+1,k} - \tilde{\rho}_{i-1,k}u_{i-1,k}}{2h} + \frac{\tilde{\rho}_{i,k+1}w_{i,k+1} - \tilde{\rho}_{i,k-1}w_{i,k-1}}{2h} = 0, \qquad (4.1)$$

$$i, = 2, 4, \ldots, N, \quad k, = 2, 4, \ldots, M,$$

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Fig. 1.

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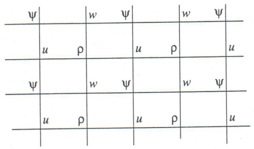

Fig. 1.

$$\tilde{\rho}_{i+1,k} = \begin{cases} \dfrac{\rho_{i+2,k}+\rho_{i,k}}{2}, & u_{i+1,k}(\rho_{i+2,k}-\rho_{i,k}) \le 0, \\ 2\dfrac{\rho_{i+2,k}\rho_{i,k}}{\rho_{i+2,k}+\rho_{i,k}}, & u_{i+1,k}(\rho_{i+2,k}-\rho_{i,k}) > 0, \end{cases} \qquad \tilde{\rho}_{i,k+1} = \begin{cases} \dfrac{\rho_{i,k+2}+\rho_{i,k}}{2}, & w_{i,k+1}(\rho_{i,k+2}-\rho_{i,k}) \le 0, \\ 2\dfrac{\rho_{i,k+2}\rho_{i,k}}{\rho_{i,k+2}+\rho_{i,k}}, & w_{i,k+1}(\rho_{i,k+2}-\rho_{i,k}) > 0. \end{cases} \quad (4.2)$$

The differential-difference equations for the velocities have the form

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\rho_{i,k}+\rho_{i+2,k}}{2}u_{i+1,k}\right) + \frac{1}{2}\left[\left(\frac{\tilde{\rho}_{i+3,k}u^2_{i+3,k}-\tilde{\rho}_{i-1,k}u^2_{i-1,k}}{4h}\right)\right. \\ &+ u_{i+1k}\frac{\tilde{\rho}_{i+3,k}u_{i+3,k}-\tilde{\rho}_{i-1,k}u_{i-1,k}}{4h} + \tilde{\rho}_{i+1,k}u_{i+1,k}\frac{u_{i+3,k}-u_{i-1,k}}{4h} \\ &\left.+\left(\frac{\tilde{\rho}_{i,k+1}w_{i,k+1}-\tilde{\rho}_{i,k-1}w_{i,k-1}}{2h} + \frac{\tilde{\rho}_{i+2,k+3}w_{i+2,k+1}-\tilde{\rho}_{i+2,k-1}w_{i+2,k-1}}{2h}\right)u_{i+1,k}\right] \\ &+ \frac{1}{4}\left[\frac{u_{i+1,k+2}-u_{i+1,k}}{2h}(\tilde{\rho}_{i,k+1}w_{i,k+1}+\tilde{\rho}_{i+2,k+1}w_{i+2,k+1})\right. \\ &\left.+\frac{u_{i+1,k}-u_{i+1,k-2}}{2h}(\tilde{\rho}_{i,k-1}w_{i,k-1}+\tilde{\rho}_{i+2,k-1}w_{i+2,k-1})\right] + \frac{p_{i+2,k}-p_{i,k}}{2h} = 0, \\ &\frac{\partial}{\partial t}\left(\frac{\rho_{i,k}+\rho_{i,k+2}}{2}w_{i,k+1}\right) + \frac{1}{2}\left[\left(\frac{\tilde{\rho}_{i,k+3}w^2_{i,k+3}-\tilde{\rho}_{i,k-1}wu^2_{i,k-1}}{4h}\right)\right. \\ &+ w_{i+1,k}\frac{\tilde{\rho}_{i,k+3}w_{i,k+3}-\tilde{\rho}_{i,k-1}w_{i,k-1}}{4h} + \tilde{\rho}_{i+1,k}w_{i,k+1}\frac{w_{i,k+3}-w_{i,k-1}}{4h} \\ &\left.+\left(\frac{\tilde{\rho}_{i+1,k+2}u_{i+1,k+2}-\tilde{\rho}_{i-1,k+2}u_{i-1,k+2}}{2h} + \frac{\tilde{\rho}_{i+1,k}u_{i+1,k}-\tilde{\rho}_{i-1,k}u_{i-1i,k}}{2h}\right)w_{i,k+1}\right] \\ &+ \frac{1}{4}\left[\frac{w_{i+2,k+1}-w_{i,k+1}}{2h}(\tilde{\rho}_{i+1,k+2}u_{i+1,k+2}+\tilde{\rho}_{i+1,k}w_{i+1,k})\right. \\ &\left.+\frac{w_{i,k+1}-w_{i-2,k+1}}{2h}(\tilde{\rho}_{i-1,k+2}u_{i-1,k+2}+\tilde{\rho}_{i-1,k}u_{i-1,k})\right] + \frac{p_{i,k+2}-p_{i,k}}{2h} + \left(\frac{\rho_{i,k}+\rho_{i,k+2}}{2}\right)g = 0. \end{aligned} \quad (4.3)$$

We eliminate $p_{i,k}$ from (4.3) by the cross difference differentiation of the equations and substitute

$$w_{i,k+1} = -\frac{\psi_{i+1,k+1}-\psi_{i-1,k+1}}{2h}, \quad u_{i+1,k} = \frac{\psi_{i+1,k+1}-\psi_{i+1,k-1}}{2h} \quad (4.4)$$

Fig. 2.

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to obtain a closed system of ordinary differential equations with the unknown functions $\psi_{ik}(t)$ and $\rho_{ik}(t)$. The boundary conditions have the form

$$\begin{aligned} \psi_{-1,k} &= \psi_{1,k} = \psi_{N+1,k} = \psi_{N+3,k} = 0, \quad k = -1, 1, 3, \ldots, M+3, \\ \psi_{i,-1} &= \psi_{i,1} = \psi_{i,M+1} = \psi_{i,M+3} = 0, \quad i = -1, 1, 3, \ldots, N+3. \end{aligned} \tag{4.5}$$

Here, the conditions $\psi_{1,k} = \psi_{N+1,k} = 0$ and $\psi_{i,1} = \psi_{i,M+1} = 0$ are important and lead to $(\mathbf{v}, \mathbf{n})|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the normal to the boundary $\partial\Omega$. The conditions $\psi_{-1,k} = \psi_{N+3,k} = 0$ and $\psi_{i,1} = \psi_{i,M+3} = 0$ do not influence the behavior of the solution inside Ω; it is only important that the quantities involved be finite.

Lemma 1. *If* $\rho_{i,k}(0) > 0$ *for all admissible i and k, then* $\rho_{i,k}(t) > 0$.

Lemma 2. *For* (4.1) *and* (4.2), *the grid analogue of relation* (3.6) *holds and*

$$\frac{d}{dt}\sum_{\Omega}\rho_{i,k} = 0, \quad \frac{d}{dt}\sum_{\Omega}\left[\rho_{i,k}\ln\left(\frac{\rho_{i,k}}{\rho_0(0)}\right)\right] \le 0. \tag{4.6}$$

Remark 3. In fact, a particular difference approximation chosen in the continuity equation determines ν in (3.6). On the density discontinuity, $|\nu|$ is proportional to $[\rho]v$, where $[\rho]$ is the jump in the density and v is the velocity amplitude. Therefore, $|\nu|$ is small if the density jump is small or the wave amplitude is small. Instead of (4.2), we can propose other approximations that lead to small $|\nu|$ and $\nu \le 0$. Therefore, the solution to the generalized problem seems to be nonunique. Physical considerations suggest the additional condition that $|\nu|$ is minimal if $\nu \le 0$. However, the density of the fluid considered in this paper ranges in a short interval (only 4%) and $|\nu|$ is automatically very small for approximation (4.2). Therefore, the minimality condition on $|\nu|$ is unimportant. However, it can be important for strongly stratified fluids.

The numerical scheme proposed is conservative, and

$$\frac{d}{dt}\left[\sum_{\Omega}\frac{\rho_{i,k}+\rho_{i+2,k}}{2}\frac{u^2_{i+1,k}}{2} + \sum_{\Omega}\frac{\rho_{i,k}+\rho_{i,k+2}}{2}\frac{w^2_{i,k+1}}{2} + \sum_{\Omega}\rho_{i,k}gz_k\right] = 0. \tag{4.7}$$

Theorem 2. *The solution to problem* (4.1), (4.3), (4.5) *exists if* $\rho_{ik}(t=0) > 0$ *and* $\tilde{H}_{\text{nonl}}(t=0) < \infty$. *The solution satisfies* $\rho_{ik}(t) > 0$ *and* $d\tilde{H}_{\text{nonl}}/dt \le 0$, *where*

$$\begin{aligned} \tilde{H}_{\text{nonl}} &= \sum_{\Omega}\frac{\rho_{i,k}+\rho_{i+2,k}}{2}\frac{u^2_{i+1,k}}{2} + \sum_{\Omega}\frac{\rho_{i,k}+\rho_{i,k+2}}{2}\frac{w^2_{i,k+1}}{2} \\ &+ \sum_{\Omega}g\left\{\rho_{i,k}\left[z_k + H\ln\left(\frac{\rho_{ik}}{\rho_0(0)}\right)\right] + H[\rho_0(z_k) - \rho_{i,k}]\right\} \ge 0. \end{aligned} \tag{4.8}$$

In (4.8), the sums are taken over interior points on the grid shown in Fig. 1 and $\tilde{H}_{\text{nonl}}$ is the discrete analogue of H_{nonl}. All the expressions under the summation signs in (4.8) are nonnegative.

All the grid functions are extended between grid points so that the resulting functions are piecewise constant. Multiplying Eqs. (4.1) and (4.3) by smooth functions and applying integration by parts, we can show that the extended solution to system (4.1), (4.3), (4.4), (4.5) satisfies (3.10), (3.12), and (3.13) when $h \longrightarrow 0$. This means that the method, if convergent, can be used to calculate not only classical but also weak solutions.

Remark 4. The uniqueness of the solution and the stability of the numerical method are not analyzed here. Note only that, in the limit of small-amplitude waves, the functional H_{nonl} in (3.3) passes into H_{lin} defined in (3.1), the solution of the nonlinear equations tends to that of the linearized equations, and the solution to the linearized equations is unique and stable, which can easily be shown.

Fig. 3.

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