

## ***Interactive comment on “Numerical investigation of stability of breather-type solutions of the nonlinear Schrödinger equation” by A. Calini and C. M. Schober***

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We thank the Referee for a careful reading of our manuscript and an opportunity to further explain our work.

Questions (Q) and Responses(R):

(Q1) Are the class of perturbations you consider inside the solutions of the NLS equation, or are they outside this class, i.e. at higher order beyond the NLS equation? Since you are using Floquet theory for NLS I assume you are just asking: Inside the class of solutions of the NLS equation you define your initial conditions as being, say,

C2479

a breather and other smaller components, right? Could these perturbations simply be viewed as other components in the Riemann spectrum? That is, could we a priori write the Riemann matrix, frequencies and phases and just get the properties of the NLS solution from this? So that the new situation is a perturbed solution of NLS that we treat as a new initial condition of NLS and we would like to find out if the breather will behave roughly physically as it would in the absence of the perturbation?

(R1) The framework of the linear stability analysis employed in this paper applies to spatially periodic perturbations of multi-phase NLS solutions as well as degenerate multi-phase solutions. In other words, it can be used to determine the linear stability type of N-gap solutions as well as their homoclinic orbits. Notice that the perturbation does not need to be in the same class of these breathers, but need only be square integrable and periodic of period  $L$ . The reason lies in the fact that the squared eigenfunctions of periodic N-gap solutions typically form a basis for the  $L^2_{\text{periodic}}$ , thus a complete set of solutions of the linearized NLS equation about the given N-gap solution. The advantage is that we can consider perturbations that are physically reasonable and sufficiently general (periodic in  $L$  with no major restrictions such as belonging to a special class of NLS solutions). The perturbed initial condition and thus the perturbed NLS solution will no longer be a finite gap one, but, in the case of neutral stability, it will remain close to the unperturbed solution (for finite time). Note that this paper also explores even more general types of perturbation: there is no available analysis for such cases, but we observe a behavior similar to the one that we can explain using the linearized analysis.

(Q2) It seems clear that you are dealing with breather solutions that have a two-by-two Riemann matrix (I think this must be a “one-mode breather” in your terminology?) and four-by-four Riemann matrix (I think this must be a “two-mode breather” in your terminology? That is, for the latter two “one-mode breathers” that have different phases. Others might have “coalesced” or have been “phase locked” in the sense of the Riemann spectrum components, right?).

C2480

(R2) This is correct, one way to interpret the breathers we consider is as degenerate 2- and 3-phase NLS solutions close to unstable Stokes waves (or plane waves). Degenerate here means that these are the result of a limiting process in which the temporal frequencies have been sent to zero. This produces spatially periodic homoclinic orbits of plane waves. See below for the spectral plane description of these solutions. Note: it is worth comparing these with the Peregrine breathers, in which both spatial and temporal frequencies have been “sent” to 0 in a limiting process, producing rational solutions in both  $x$  and  $t$ . When sending the temporal frequencies to 0, the solution remains periodic in  $x$  but develops sech-type decay in time (typical of homoclinic solutions).

(Q3) Perhaps you have thought about where these solutions lie in the complex lambda plane? Would this help me understand more quickly, what you have done?

(R3) The Floquet spectrum of the breathers considered is very simple: one imaginary band, and a finite number of complex (imaginary) double points along the band. The linear stability analysis is applied for the ones with one and two imaginary double points. Note that this is the same spectrum as the underlying Stokes wave to which these breathers are homoclinic. The reason is that they all belong to the same level set (thus they belong to the same isospectral set).

(Q4) I assume that for neutral stability you might also mean, practically speaking, “physically realizable” or “physically stable”? Therefore, neutral stability is a good thing to find?

(R4) Neutral stability is the absence, in the solution of the linearization, of any exponentially-in-time growing modes. Physically this means that a small perturbation of the initial condition will remain small for finite time: nothing can be said for longer time intervals using linear analysis, it is our hope that we can say something stronger along the line or orbital stability, as the numerics seem to suggest, but our analysis is not yet there. If you had linear stability, that would mean that a small perturbation would

C2481

decay to become close to zero over a finite interval of time, a rather strong property. I would argue that neutral stability is a sufficient criterion for a solution to be “physically realizable”: if we introduce a small noise/change initially and the solution keeps close to the unperturbed solution for some finite time, that in practice means that I would observe that solution in the course of an experiment (which invariably will introduce some noise). It would be too strong to demand that any initial noise die down, but reasonable to ask that such noise stays “controlled”.

(Q5) You use the word “saturated” but I am confused as to what this means. Is there a way to understand “saturated” from the point of view of the lambda plane?

(R5) The lambda plane picture does not explain the saturation phenomenon directly, but here is a way to perhaps make the connection. Take the underlying unstable Stokes wave characterized by an imaginary spine of continuous spectrum and  $M$  imaginary double points along this spine. Each double point corresponds to an unstable mode (meaning: if we linearize the NLS about the Stokes wave, then we would find a Fourier mode which grows exponentially in time with growth rate related to the location of the double point). Now, for each of those “instabilities”, using a Backlund transformation we can produce a breather that limits in forward and backward time to the Stokes wave along the corresponding unstable eigenspace. Now, when we linearize the NLS about this breather solution, we discover that the previously unstable mode is no longer unstable, but has become neutral! Mathematically this means that what was before a term looking like  $\exp(\alpha t)$  with real part of  $\alpha$  different from zero (causing exponential growth in the solution of the linearization), it has now become  $\exp(i \alpha t)$  with  $\alpha$  real, no longer producing exponential growth in time. This phenomenon is known as “saturation” of the instability: (very) loosely speaking, you construct the homoclinic orbit corresponding to the linear instability, and you have thus “used up” that instability.

(Q6) In the past you have used “spines” to characterize breather solutions and even inferred whether you have homoclinic solutions or not from the spines. Can you tell me something about the spines in the present case? Aren't we just getting a new

C2482

spectrum that is perturbed by smaller components, and then asking: What happens to the breathers when we add a small perturbation to them that is fully characterized by the Riemann spectrum (parameters) of the theta functions?

(R6) The Floquet spectrum of the perturbed breather will look like a perturbed band (close to the imaginary band of spectrum) and an infinite number of small spines originating from what were the imaginary double points and all the real double points. It is an infinite-gap solution close in spectrum to the spectral configuration of the Stokes wave and of its homoclinic orbit. What we can entirely predict by the Riemann spectrum of the underlying breather (which is a degenerate theta function) is its linear stability. This is because its linear stability is completely encoded in the squared eigenfunctions, constructed out of the Baker-Akhiezer eigenfunction for that breather. What we still do not have, but would be great to have, is some conclusion about orbital stability: i.e. some criterion that would say “the perturbed solution remains close to one of the family of breathers for all times”. Linearized analysis does not provide this answer, but it is a start. Notice that, for the cases when we can predict neutral stability, the numerics do suggest something a bit stronger, along the lines of orbital stability.

As suggested, we will modify the abstract and include a short summary to clarify the implications of our work.

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