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Discussions

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Collisions of two breathers at the surface of deep water

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Abstract

We present results of numerical experiments on long time evolution and collisions of breathers (which correspond to envelope solitons in the NLSE approximation) at the surface of deep ideal fluid. The collision happens to be nonelastic. In the numerical experiment it can be observed only after many acts of interactions. This supports the hypothesis of “deep water nonintegrability”. The experiments were performed in the framework of the new and refined version of the Zakharov equation free of nonessential terms in the quartic Hamiltonian. Simplification is possible due to exact cancellation of nonelastic four-wave interaction.

1 Introduction

Theory of weakly nonlinear waves on shallow water is a nursery for several completely integrable models. Among them the famous KdV and KP equations (Gardner et al., 1967; Kadomtsev and Petviashvili, 1973; Zakharov and Shabat, 1979), Boussinesq equation (Zakharov, 1974), Kaup system (Kaup, 1975). Detailed study of these integrable systems has not only theoretical, but also practical importance. Recently A. Osborne has shown (Osborne, 2010) that representation of solutions of KP equations in form of Jacobi theta-functions is a very efficient and economic way of analyzing of experimental data for long waves in coastal area.

Now the fundamental question appears – what can be done in the case of deep fluid? So far only one integrable model on deep water is known. This is the focusing Nonlinear Schrodinger equation describing weakly-nonlinear quasimonochromatic wave trains (Zakharov, 1968; Zakharov and Shabat, 1972). Exact solutions of this equation can be also given by theta-functions (Belokolos et al., 1994). They are actively used now for determination of freak wave statistics (Osborne, 2010). However the NLSE has a limited area of application and hardly can be applicable to many experimental situations.

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Hopes that the exact Euler equation for potential flow on deep water with free surface in the presence of gravity is integrable appeared in 1994 when two of us (Dyachenko and Zakharov, 1994) established that coefficient of scattering matrix connecting asymptotic at $t \to \pm\infty$ states of wave field, corresponding to inelastic four-wave processes and governed by resonant conditions

$$k + k_1 = k_2 + k_3 \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3},$$

where

$$\omega_k = \sqrt{g|k|}$$

in 1-D geometry is identically zero.

However this cancellation is just a weak necessary condition for integrability far from being sufficient. For integrability in its "strong sense" we need cancellation in all orders of perturbation theory (see Zakharov and Schulman, 1991). However in Dyachenko et al. (1995) it was shown that not all members of five-wave scattering matrix are zero, thus we can hope only integrability in some "weak sense". We will not discuss here this subject having a "strong mathematical flavor ".

Meanwhile efficient methods for numerical simulations of exact Euler equation were been developed during last decade, massive numerical experiments were also performed. Again, some of them can be considered as certain indication of integrability.

In the frame of NLSE approximation we have an exact solution – envelope soliton. Do such solutions exist in the exact Euler equation? If the system is not-integrable, the soliton exists only during a finite time, then it must lose its energy due to radiation in backward direction (Zakharov and Kuznetsov, 1998). In the nonintegrable MMT model this backward radiation is a very strong effect leading to formation of "abnormal" weak turbulent spectrum (Rumpf et al., 2009). However, in our experiments on propagation of steep envelope solitons in the frame of Euler equation we did not trace a slightest backward radiation (Dyachenko and Zakharov, 2008). The soliton persistently existed during thousands of their periods.

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In this article we present new numerical results shedding some light on integrability of the deep-water hydrodynamics. We study collision of breathers (solitons) in the frame of new derived approximate equation applicable for small amplitude waves with any spectral band width. Actually, this is what is called "Zakharov equation" (see Zakharov, 1968) improved by implementation of additional canonical transformation to the Poincare normal form. This transformation is possible only due to still the mysterious fact of four-wave interaction cancellation.

The new equation (described in details in Dyachenko and Zakharov, 2011, 2012) is very convenient for numerical simulations. It has nice solitonic solution which can not be so far found analytically.

In this paper we study collision of such solitons and show that this collision is nonelastic. But one can see it after multiple collisions only. We can interpret this fact as a numerical proof of nonintegrability at least for this "refined Zakharov equation"[1].

2 Compact equation

A one-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is described by the following set of equations:

$$\phi_{xx} + \phi_{zz} = 0 \quad (\phi_z \to 0, z \to -\infty),$$

$$\eta_t + \eta_x \phi_x = \phi_z \Big|_{z=\eta}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \Big|_{z=\eta};$$

here $\eta(x,t)$ – is the shape of a surface, $\phi(x,z,t)$ – is a potential function of the flow and g – is a gravitational acceleration. As was shown in Zakharov (1968), the variables

[1] Part of the numerical results were put in (Dyachenko et al., 2012).

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$\eta(x,t)$ and $\psi(x,t) = \phi(x,z,t)\Big|_{z=\eta}$ are canonically conjugated, and satisfy the equations

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta} \qquad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}.$$

Here Hamiltonian can be written as infinite series (see Zakharov, 1968):

$$H = \frac{1}{2}\int g\eta^2 + \psi\hat{k}\psi \mathrm{d}x - \frac{1}{2}\int\{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta \mathrm{d}x + \frac{1}{2}\int\{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}\mathrm{d}x + \ldots \quad (1)$$

It this article we consider Hamiltonian up to the fourth order. In the articles (Dyachenko and Zakharov, 2011, 2012) we applied canonical transformation to the hamiltonian variables ψ and η to introduce normal canonical variable $b(x,t)$. This transformation explicitly exploits vanishing of four-wave interaction and possibility to consider surface waves moving in the same direction. For this variable $b(x,t)$ Hamiltonian (Eq. 1) acquires nice and elegant form[2]:

$$\mathcal{H} = \int b^*\hat{\omega}_k b \mathrm{d}x + \frac{1}{2}\int\left|\frac{\partial b}{\partial x}\right|^2\left[\frac{i}{2}\left(b\frac{\partial b^*}{\partial x} - b^*\frac{\partial b}{\partial x}\right) - \hat{K}|b|^2\right]\mathrm{d}x. \quad (2)$$

In K-space Hamiltonian has the form:

$$\mathcal{H} = \int \omega_k |b_k|^2 \mathrm{d}k + \frac{1}{2}\int \tilde{T}^{k_3k_4}_{k_1k_2} b^*_{k_1} b^*_{k_2} b_{k_3} b_{k_4} \delta_{k_1+k_2-k_3-k_4} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \quad (3)$$

[2]There was the misprint in the articles (Dyachenko and Zakharov, 2011, 2012), coefficient for quartic term in the hamiltoian must be $\frac{1}{2}$ instead of $\frac{1}{4}$.

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Here

$$\widetilde{T}^{kk_1}_{k_2k_3} = \frac{\theta(k)\theta(k_1)\theta(k_2)\theta(k_3)}{8\pi}\left[(kk_1(k+k_1)+k_2k_3(k_2+k_3))\right.$$
$$\left.-(kk_2|k-k_2|+kk_3|k-k_3|+k_1k_2|k_1-k_2|+k_1k_3|k_1-k_3|)\right], \quad (4)$$

$$\theta(k) = \begin{cases} 0, & \text{if } k \le 0; \\ 1, & \text{if } k > 0. \end{cases}$$

The Fourier transform is defined as follow:

$$b(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} b_k e^{ikx}\mathrm{d}x$$

$b(x)$ – can be analytically continued to $x+iy$, $y>0$. Motion equation for b_k should be understood as follow:

$$i\frac{\partial b}{\partial t} = \widehat{P}^{+}\frac{\delta\mathcal{H}}{\delta k_k^*},$$

here $\widehat{P}^{+}$ – projection operator to the upper half-plane.

$$\widehat{P^{+2}} = \widehat{P}^{+} = \frac{1}{2}(1 - i\widehat{H}).$$

This operator in the consequence of θ-functions (positive k only) in Eq. (3). Summarizing we consider self-consistent system of waves propagating in the same direction.

Corresponding equation of motion is the following:

$$i\frac{\partial b}{\partial t} = \widehat{\omega}_k b + \frac{i}{4}\widehat{P}^{+}\left[b^*\frac{\partial}{\partial x}(b'^2) - \frac{\partial}{\partial x}(b^{*\prime}\frac{\partial}{\partial x}b^2)\right]$$
$$-\frac{1}{2}\widehat{P}^{+}\left[b\cdot\widehat{K}(|b'|^2) - \frac{\partial}{\partial x}(b'\widehat{K}(|b|^2))\right], \quad (5)$$

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or in K-space

$$i\frac{\partial b_k}{\partial t} = \omega_k b_k + \int \widetilde{T}_{kk_1}^{k_2k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3. \quad (6)$$

3 Breathers and numerical simulation of its collisions

Breather is the localized solution of Eq. (5) of the following type:

$$b(x,t) = B(x - Vt)e^{i(k_0 x - \omega_0 t)}, \quad (7)$$

where k_0 is the wavenumber of the carrier wave, V is the group velocity and ω_0 is the frequency close to ω_{k_0}. In the Fourier space breather can be written as follow:

$$b_k(t) = e^{-i(\Omega + Vk)t}\phi_k, \quad (8)$$

where Ω is close to $\frac{\omega_{k_0}}{2}$.

For ϕ_k the following equation is valid:

$$(\Omega + Vk - \omega_k)\phi_k = \int \widetilde{T}_{kk_1}^{k_2k_3} \phi_{k_1}^* \phi_{k_2} \phi_{k_3} \delta_{k+k_1-k_2-k_3} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3. \quad (9)$$

One can treat ϕ_k as pure real function of k.

To solve Eq. (9) one can use Petviashvili iteration method (n – is the number of iteration):

$$(\Omega + Vk - \omega_k)\phi_k^{n+1} = M^n \int \widetilde{T}_{kk_1}^{k_2k_3} \phi_{k_1}^{*\,n} \phi_{k_2}^n \phi_{k_3}^n \delta_{k+k_1-k_2-k_3} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3. \quad (10)$$

Petviashvili coefficient M^n is the following:

$$M^n = \left[\frac{< \phi_k^n (\Omega + Vk - \omega_k)\phi_k^n >}{< \phi_k^n \int \widetilde{T}_{kk_1}^{k_2k_3} \phi_{k_1}^{*\,n} \phi_{k_2}^n \phi_{k_3}^n \delta_{k+k_1-k_2-k_3} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 >} \right]^{\frac{3}{2}}.$$

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Below we present a typical numerical solution of Eq. (9). Calculations were made in the periodic domain 2π with carier wavenumber $k_0 \sim 64$, $V = 1/16$ and $\Omega = 4.01$. In the Figs. 1, 2, 3 one can see real part of $b(x)$, modulus of $b(x)$ and Fourier spectrum of $b(x)$.

To analyze the question about integrability of the Eq. (5) one can consider collisions of breathers. It might be elastic or nonelastic. To study breathers collision we performed the following numerical simulation:

- As initial condition we have used two breathers separated in space (distance was equal to π).
- First breather has the following parameters: $\Omega_1 = 4.01$, $V_1 = 1/16$. Carrier wave number appears to be ~ 64.
- For the second breather: $\Omega_2 = 4.51$, $V_2 = 1/18$. Carrier wave number appears to be ~ 81.

This initial condition is shown in Fig. 4. Its Fourier spectrum is shown in Fig. 5. After time $\frac{\pi}{(V_1 - V_2)} \simeq 452.4$ breathers collides. In the Fig. 6 one can see breathers at the moment of collision ($t = 452.4$). Fourier spectrum of two breathers at $t = 452.4$ is shown in Fig. 7. And finally we show the picture of two breathers after 100 collisions at $t \sim 88\,000$ when they separated again at distance $\simeq \pi$.The initial condition and state after 100 breather collisions are shown in Fig. 8. Fourier spectrum of that is given in Figs. 9. One can see radiation after collisions in Fig. 10 that shows the zoomed profile of $|b(x)|$. During numerical simulation the total energy was conserved up to ninth digit after decimal point. So, the simulation demonstrates that after multiple collisions of breathers there appears radiation. It points on nonelastic collisions and nonintegrability of the Eq. (5).

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4 Conclusions

We see that individual breathers are not differ from NLSE solitons qualitatively. We have studied numerically interaction of two solitons with different values of carrier wave lengths. Interaction of such breathers cannot be described by the NLSE even approximately.

Interaction of such solitons happens to be nonelastic. This experimental fact requires additional study to prove nonintegrability analytically.

This new Eq. (5) can be generalized for the "almost" 2-D waves, or "almost" 3-D fluid. When considering waves slightly inhomogeneous in transverse direction, one can think in the spirit of Kadomtsev–Petviashvili equation for Korteveg–de-Vries equation, namely one can treat now frequency ω_k as two dimensional, ω_{k_x,k_y}, while leaving coefficient $\widetilde{T}^{kk_1}_{k_2k_3}$ not dependent on y. b now depends on both x and y:

$$\mathcal{H} = \int b^* \widehat{\omega}_{k_x,k_y} b \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int |b'_x|^2 \left[\frac{i}{2}(bb'^*_x - b^* b'_x) - \widehat{K}_x |b|^2\right] \mathrm{d}x\mathrm{d}y. \quad (11)$$

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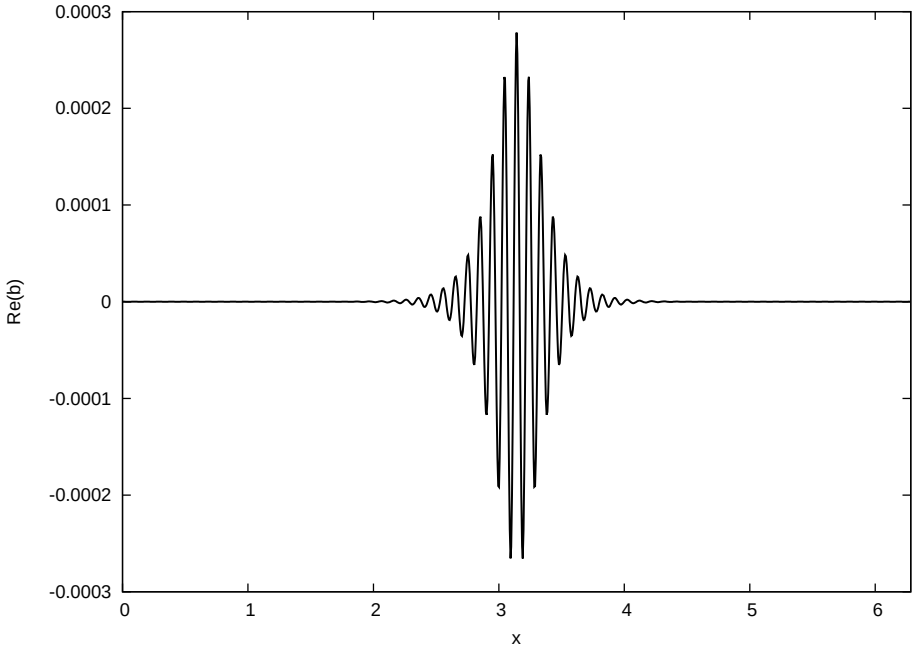

Fig. 1. Real part of $b(x)$ with $V = 1/16$ and $\Omega = 4.01$.

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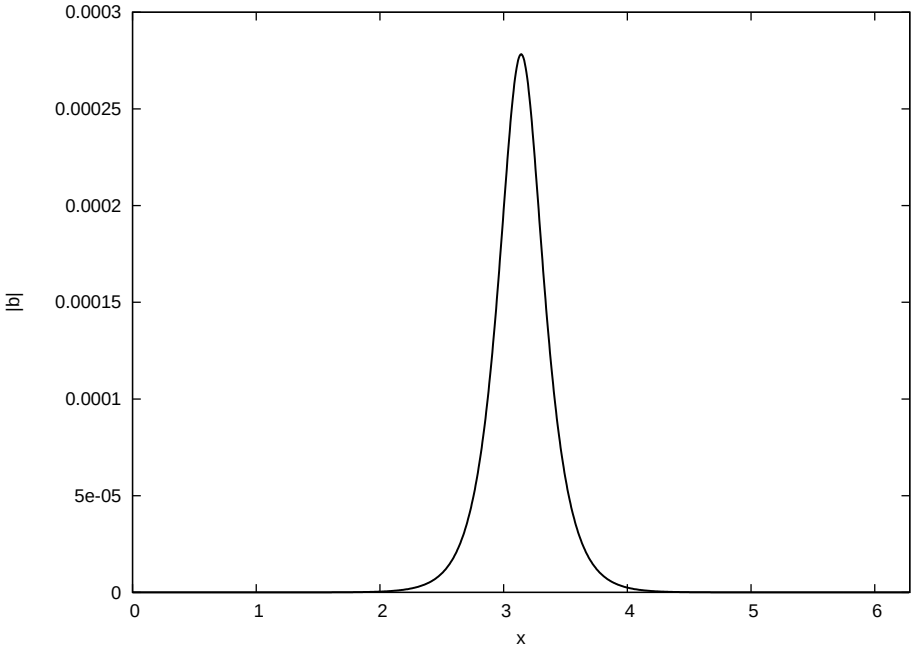

Fig. 2. Modulus of $b(x)$ with $V = 1/16$ and $\Omega = 4.01$. (Recall envelope in the Nonlinear Schredinger Equation.)

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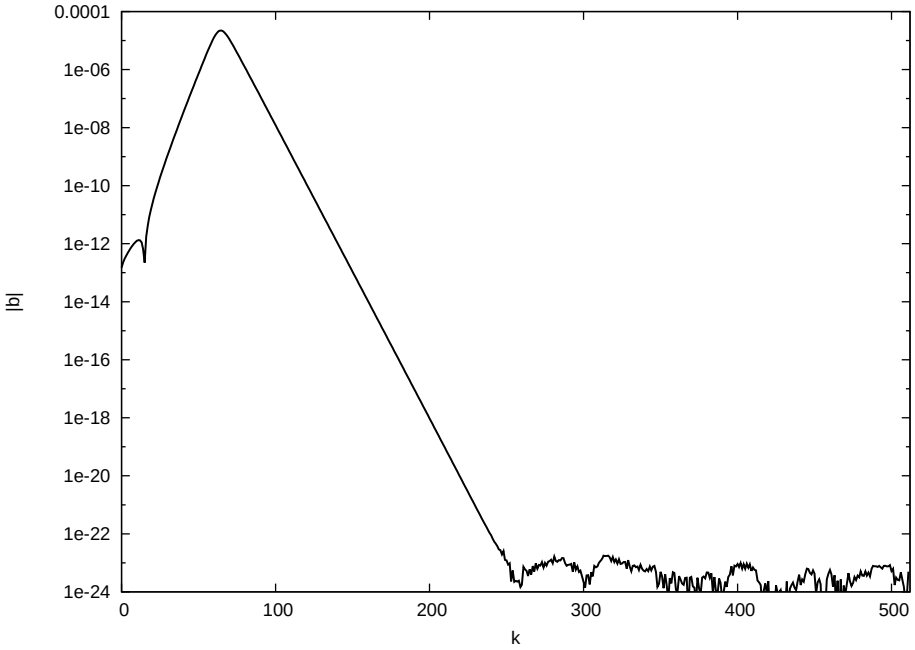

Fig. 3. Spectrum of $b(x)$ with $V = 1/16$ and $\Omega = 4.01$.

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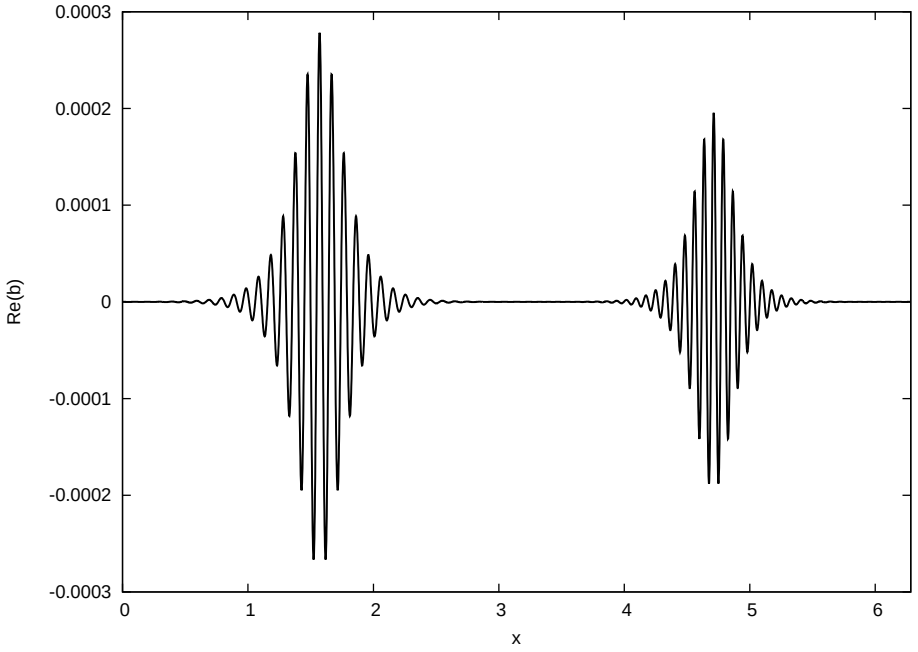

Fig. 4. Initial condition with two breathers.

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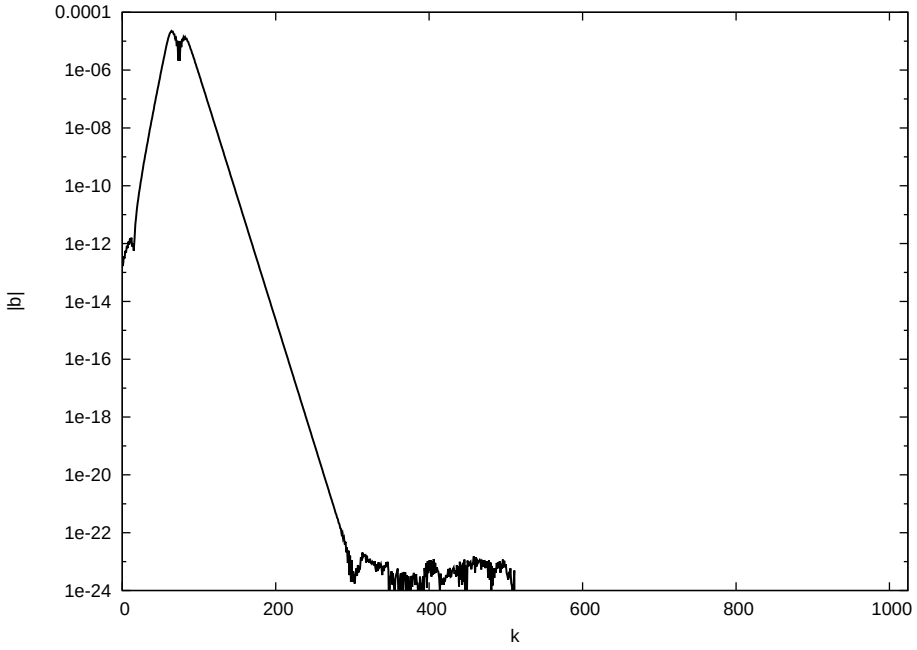

Fig. 5. Fourier spectrum of two breathers.

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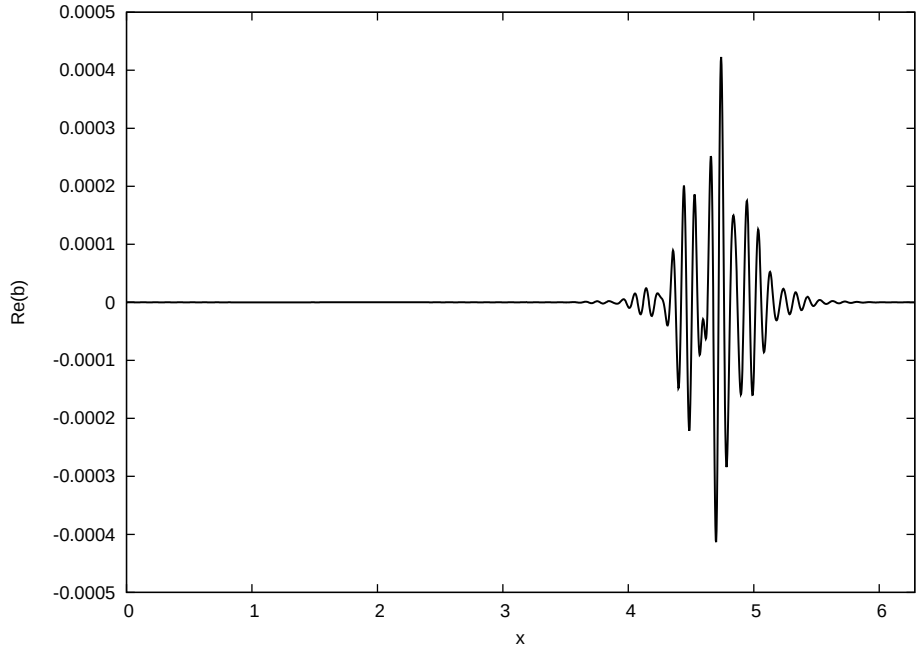

Fig. 6. Two breathers collides.

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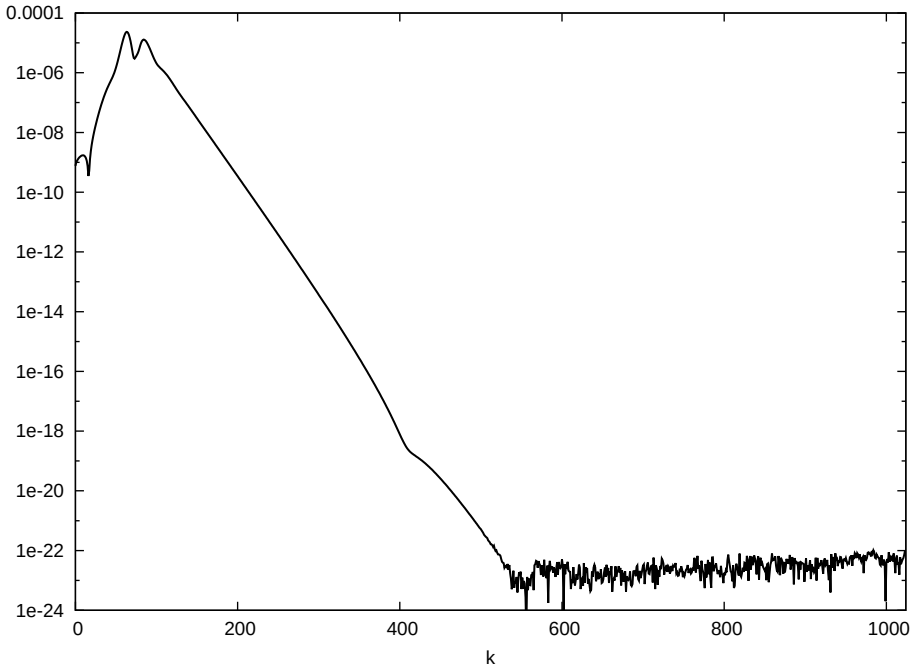

Fig. 7. Fourier spectrum at the moment of collision.

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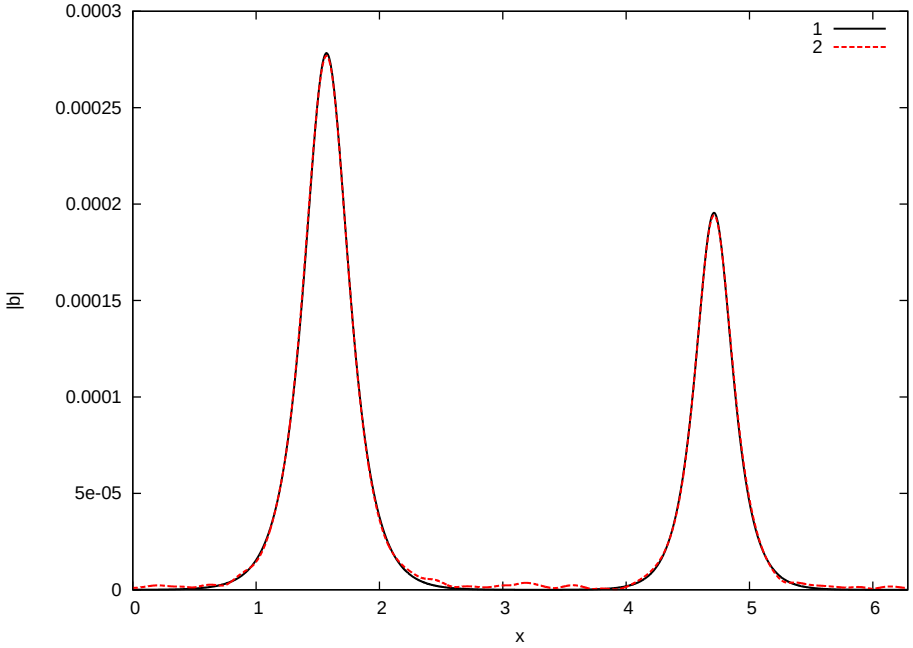

Fig. 8. Modulus of $b(x)$ for two point of time. Solid line corresponds the initial statement ($t = 0$), dashed line corresponds to the state after 100 breather collisions ($t \sim 88\ 000$).

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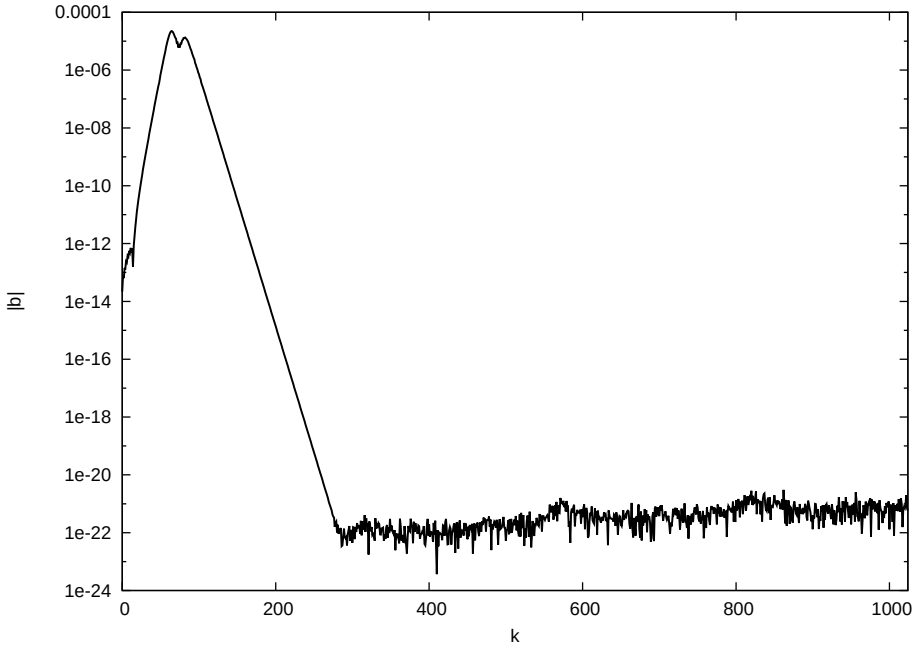

Fig. 9. Fourier spectrum after collision.

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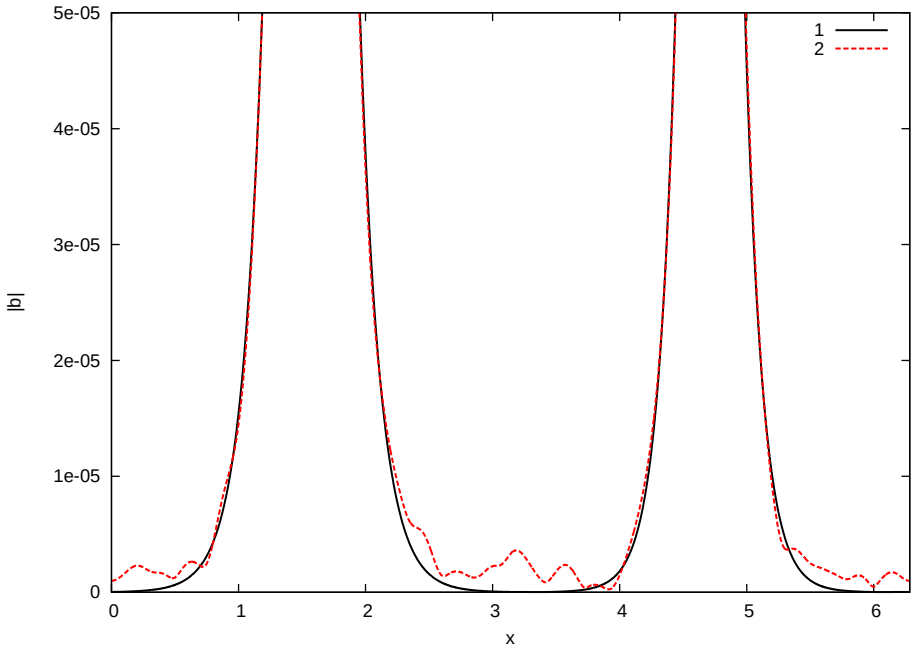

Fig. 10. Modulus of $b(x)$ for two point of time. Solid line corresponds the initial statement ($t = 0$), dashed line corresponds to the state after 100 breather collisions ($t \sim 88\ 000$).

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