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Buried and displaced: moving characteristics of building fragments in debris flows

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Introduction

This file provides the supplementary derivation for Section 3.3 (analytical solution) of the main text. Section S1 to S3 detail the solutions of the block velocity and displacement in the analytical model, as well as the classification of different solutions in the model.

S1: Solution of block velocity

The solution of Model I is a synthesis of Model II and Model III, including the solution form and its classification. We start with the simplified Model II and Model III.

S1.1 Model II: considering only dynamic drag force

Note this section is also detailed in the main text as Section 3.4.2 An example of the model solution.

When only the dynamic drag force is considered, the dimensionless momentum Equation (16) can be expressed as:

$$-\frac{d\left(\Delta v^*\right)}{dt^*} = D^*\left(\Delta v^*\right)^2 Sgn\left(\Delta v^*\right) + G^* \tag{S1}$$

Equation (S1) is an ordinary differential equation whose solution form depends on the positive or negative coefficients in front of its variables. Since D^* is always positive, the solution form of Equation (S1) is primarily determined by the sign of G^* and *Sgn*(). That *Sgn*() reflects the relative magnitude of the debris flow basal velocity and block velocity, during block sliding. If the block velocity exceeds the debris-flow basal velocity, the sign of *Sgn*() will change. G^*, which relates to relative density and equivalent acceleration difference, is independent of velocity difference changes during motion. In our experiments, all G^* exhibit negative values (G^*<0), necessitating that the classification paradigm be exclusively contingent upon scenarios where G^*<0.

The block's movement process can be divided into two states:

State 1: Δv^*<0. This state occurs during the initial states of debris flow displacing blocks, where both velocities are decreasing. Throughout this process, the block velocity persistently surpasses the debris-flow basal velocity, so the block velocity solution can be expressed:

$$v_b^* = \frac{1}{n}\left(1-\frac{G_d^*}{2}t^*\right)-\sqrt{-\frac{G^*}{D^*}}\tan\left[-\sqrt{-G^*D^*}t^*+\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{-\frac{D^*}{G^*}}\right)\right] \quad \text{(S2)}$$

According to Equation (S2), the dimensionless time (t_1^*) when debris flow and block attain the same velocity ($v_b^* = v_d^* = \frac{1}{n}\left(1-\frac{G_d^*}{2}t^*\right)$) is given by:

$$t_1^* = \frac{1}{\sqrt{-G^*D^*}}\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{-\frac{D^*}{G^*}}\right) \quad \text{(S3)}$$

From Equation (S1), we can get the dimensionless stop time of debris flow is $2/G_d^*$. If $t_1^*>2/G_d^*$, the time that block's velocity is the same as flow velocity is larger than the time flow stop time, but debris flow has already stopped, so the block's velocity is always larger debris flow's basal-velocity in this condition. If $t_1^*<2/G_d^*$, the block's velocity will lower than debris-flow basal velocity at t_1^* (Figure 9c), then *Sgn*() changed it's sign (from negative to positive) and the block velocity solution after t_1^* will have another form, the changed solution as expressed in State 2.

State 2: $\Delta v^*>0$. This state occurs after the time (t_1^*) and block velocity is lower than debris-flow basal velocity, where the drag force acts as driving force to block. The solution can be expressed:

$$v_b^* = \frac{1}{n}\left(1-\frac{G_d^*}{2}t^*\right)-\sqrt{-\frac{G^*}{D^*}}\tanh\left[\sqrt{-G^*D^*}t^*-\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{G^*}}\right)\right] \qquad (S4)$$

The right side of Equation (S4) is the combination of debris-flow basal velocity ($v_d^*=\frac{1}{n}\left(1-\frac{G_d^*}{2}t^*\right)$) and a hyperbolic tangent function (y=tanh(x)), as x increases, tanh(x) is infinitely close to 1. Therefore, with the increase of dimensionless time t_1^*, the dimensionless block velocity v_b^* tends to an asymptote, which is parallel to the dimensionless debris-flow basal velocity curve, and the asymptote is higher than debris-flow basal velocity curve, their difference is $\sqrt{-\frac{G^*}{D^*}}$ (Figure 9c). The asymptote is given:

$$v_b^*|_{asymptote}=\frac{1}{n}\left(1-\frac{G_d^*}{2}t^*\right)-\sqrt{-\frac{G^*}{D^*}} \qquad (S5)$$

Equation (S1)-(S5) give the block velocity time-history curve of two states. $G^*<0$ means debris flow decelerates slower than the block. If the block velocity can reach debris-flow basal velocity before debris-flow stop time T^* (T^* is solve from Equation (S4) when $v_b^*=0$. Equation (S4) is a transcendental equation, and T^* requires an approximate solution to be obtained using the least squares method), the velocity difference ($v_b^*-v_d^*$) will decrease continuously but not over an upper limit value $\sqrt{-\frac{G^*}{D^*}}$. The existence of this limit value is due to dimensionless drag force D^*, which plays a role of resistance after t_1^* and limits the increase of velocity difference: the larger the velocity difference, the larger the resistance. So that the velocity difference does not grow indefinitely but is limited to a range.

S1.2 Model III: considering only earth pressure

When only the earth pressure is considered, the dimensionless momentum Equation (17) can be expressed:

$$-\frac{d\left(\Delta v^*\right)}{dt^*} = K^* Sgn\left(\Delta v^*\right) + G^* \tag{S6}$$

For governing Equation (S6) of Model III, it's right side only consists of constant term $K^*Sgn(\Delta v^*)$ and G^*. The cases classification of Model III is based on the relative magnitude of K^* and $|G^*|$. Regardless of the relative magnitudes of K^* and $|G^*|$, the block's velocity curve will intersect with the velocity curve of debris flow. In all cases, the solution in state 1 can be expressed:

$$v_{\mathrm{b}}^* = \frac{1}{n}\left(1 - \frac{G_{\mathrm{d}}^*}{2} t^*\right) + \left(-K^* + G^*\right) t^* + \left(m - \frac{1}{n}\right) \tag{S7}$$

In state 2, the relative magnitude of K^* and $|G^*|$ determines the form of the solution. If the earth pressure can compensate for the deceleration difference between block and debris flow ($K^*>|G^*|$), then the block will keep the same velocity with debris flow until the end of the movement. The solution can be expressed:

$$v_{\mathrm{b}}^* = \frac{v_{\mathrm{d}}^*}{n} = \frac{1}{n}\left(1 - \frac{G_{\mathrm{d}}^*}{2} t^*\right) \tag{S8}$$

The velocity profile for State 2 provided here is not obtained by directly solving Equation (S7), but rather derived from analyzing the solution process. Section S2 explains in detail why $K^*>|G^*|$ ensures consistent velocities between the block and debris-flow basal velocity.

When $K^*<|G^*|$, the solution can be expressed:

$$v_b{}^* = \frac{1}{n}\left(1 - \frac{G_d{}^*}{2}t^*\right) + \left(K^* + G^*\right)t^* - \left(m - \frac{1}{n}\right)\frac{K^* - G^*}{K^* + G^*} \tag{S9}$$

The explanation for the phenomenon of earth pressure causing the block velocity to be the same as the velocity of the debris flow will be further explored in Section S2. It is important to note that in the absence of drag force, the block motion does not have an asymptote during state 2.

S1.3 Model I: considering both dynamic drag force and earth pressure

Model I takes into account both dynamic pressure and earth pressure. Its dimensionless momentum equation is given by Equation (10).

Similarly, the positive or negative of the coefficients in front of the variables determines the form of solutions. Equation (S1) and Equation (10) differs in the presence of dimensionless earth pressure $K^*Sgn(\Delta v^*)$, which determines if the block velocity matches the debris flow velocity. Here, K^* denotes the difference between the passive earth pressure and active earth pressure. Dimensionless dynamic force D^* remains positive, and it has no influence on solution's form. Solution form of Equation (S1) is primarily determined by the sign of G^* and *Sgn*(). But the point is that in Equation (S1) of Model II, the constant term consists only of G^*, whereas $K^*Sgn(\Delta v^*)$ and G^* together form the constant term in Equation 10 of Model I. This implies that G^* and K^* complement each other, with the earth pressure K^* counteracting the tendency of the block to detach from the debris flow, thereby making the block velocity closer to the velocity of debris flow.

The movement process of block can also be divided into two parts:

State 1: $\Delta v^*<0$. Both velocities are decreasing. The solution can be expressed:

$$v_{\mathrm{b}}^* = \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)+\sqrt{\frac{K^*-G^*}{D^*}}\tan\left[-\sqrt{\left(K^*-G^*\right)D^*}t^*+\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{K^*-G^*}}\right)\right] \quad \text{(S10)}$$

According to Equation (S10), the time (t_1^*) when debris flow and block have the same velocity is given:

$$t_1^* = \frac{1}{\sqrt{\left(K^*-G^*\right)D^*}}\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{K^*-G^*}}\right) \quad \text{(S11)}$$

The condition for the block velocity to be lower than the debris-flow basal velocity is $t_1^*<2/G_{\mathrm{d}}^*$. Otherwise, the block velocity will always be higher than the debris flow velocity. This condition is common to both models.

State 2: $\Delta v^*>0$. When block velocity and debris-flow basal velocity are the same, K^* determines whether the block velocity will be lower or be the same with the flow velocity. If $K^*>G^*$, this means that the earth pressure K^* can compensate for the effects of acceleration differences G^*. Once the block has the same velocity with the basal velocity of debris flow, the dynamic drag force D^* has no contribution to block movement. The block velocity will remain the same as the debris-flow basal velocity under the effects of earth pressures, as if the block is integrated into the debris flow. So after t_1^*, block velocity can be expressed:

$$v_{\mathrm{b}}^* = v_{\mathrm{d}}^* = \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right) \quad \text{(S12)}$$

Refer to the explanation in Section S2, it follows the same principle as Model Ⅲ. It is the same as debris-flow basal velocity. From Figure 9a, the point where the block velocity curve intersects with the debris flow velocity has a sudden slope change, and this point is the extreme point of the block velocity. The reason is that the model assumes that the direction of earth pressure is affected by the relative velocity of block and debris flow.

If $K^*<|G^*|$, the earth pressure K^* is not sufficient to compensate the effects of acceleration difference G^*. The block velocity will exceed debris flow velocity and tends to an asymptote. The block velocity can be expressed:

$$v_{\mathrm{b}}^* = \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)-\sqrt{-\frac{K^*+G^*}{D^*}}\tanh\left[\sqrt{-\left(K^*+G^*\right)D^*}t^*-\sqrt{-\frac{K^*+G^*}{K^*-G^*}}\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{K^*-G^*}}\right)\right] \quad \text{(S13)}$$

The asymptote:

$$v_{\mathrm{b}}^* = \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)-\sqrt{-\frac{K^*+G^*}{D^*}} \quad \text{(S14)}$$

The difference between the flow front velocity and block velocity of part II in Model I can be represented as $\sqrt{-\frac{K^*+G^*}{D^*}}$. However, since the Model II solely considers dynamic drag force, the difference increases to $\sqrt{-\frac{G^*}{D^*}}$. This illustrates how the earth pressure brings the block velocity's asymptote in proximity to the debris flow's velocity, consequently making it challenging for the block to disengage from the debris flow. Additionally, when K^* surpasses $|G^*|$, the block integrates with the debris flow as their velocities equalize.

S2: Derivation for scenario where block and debris flow having the same velocity in Model I and Model III

This section explains through a model-solving approach why, in State 2, the block velocity consistently matches the debris flow velocity when $K^*>|G^*|$. Model I and Model III are identical in their solving principles. Therefore, this section uses the more complex Model I as an example for solving and analysis.

The momentum equation in State 1 of Model I can be expressed:

$$-\frac{d\left(\Delta v^*\right)}{dt^*}=-D^*\left(\Delta v^*\right)^2-K^*+G^* \tag{S15}$$

where $\Delta v^*=\frac{v_{\mathrm{d}}{}^*}{n}-v_{\mathrm{b}}{}^*$, $v_{\mathrm{d}}{}^*=1-\frac{G_{\mathrm{d}}{}^*}{2}t^*$. Then the block velocity can be expressed：

$$v_{\mathrm{b}}{}^*=\underbrace{\frac{1}{n}\left(1-\frac{G_{\mathrm{d}}{}^*}{2}t^*\right)}_{\mathrm{A}}\underbrace{-\sqrt{\frac{K^*-G^*}{D^*}}\tan\left[-\sqrt{\left(K^*-G^*\right)D^*}t^*+\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{K^*-G^*}}\right)\right]}_{\mathrm{B}} \tag{S16}$$

where A represents the debris-flow basal velocity $v_{\mathrm{d}}{}^*$, at the junction point $t_1{}^*$ of State 1 and State 2, where debris flow and block have the same velocity ($v_{\mathrm{b}}{}^*$=A, B=0).

$$t_1^*=\frac{1}{\sqrt{\left(K^*-G^*\right)D^*}}\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{K^*-G^*}}\right) \tag{S17}$$

If the block velocity tends to be lower than the debris-flow basal velocity ($\Delta v^*>0$), the governing equation can be expressed:

$$-\frac{d\left(\Delta v^*\right)}{dt^*}=D^*\left(\Delta v^*\right)^2+K^*+G^* \tag{S18}$$

The block velocity is given:

$$v_b^* = \underbrace{\frac{1}{n}\left(1-\frac{G_d^*}{2}t^*\right)}_{\text{A}} - \underbrace{\sqrt{\frac{K^*+G^*}{D^*}}\tan\left[-\sqrt{\left(K^*+G^*\right)D^*}t^* + \sqrt{\frac{K^*+G^*}{K^*-G^*}}\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{K^*-G^*}}\right)\right]}_{\text{C}} \quad \text{(S19)}$$

When $t^*=t_1^*$, the block velocity is equal to the debris-flow basal velocity, and C=0. However, after t_1^*, C increases with time, resulting in an increase in the block velocity relative to the debris-flow basal velocity ($v_b^*>$A). This violates the initial assumptions of the governing Equation (S18). Therefore, it can be assumed that the block velocity becomes higher than the debris flow velocity after t_1^*, causing the governing equation to return to Equation (S15).

At t_1^*, B equals to 0 and remains greater than 0 after t_1^*. If v_b^* is greater than A, then the block velocity would be higher than the debris flow velocity, which contradicts the assumptions of Equation (S15) ($v_b^*<$A) too. We conclude that in the case of $K^*>|G^*|$, once the block and debris flow reach the same velocity, the block will continue moving at the same velocity with the debris flow until the end of the movement.

S3: Solution of block displacement

The displacement of the block is obtained by integrating its velocity. Due to the different solution forms of velocities in the two states, the block displacement can be divided into two states as well. The ratio of the block displacement L^* to the length of the debris flow deposition S^* represents the relative position of the block within debris flow deposition. The integration process is relatively straightforward, and only the integration results are provided here. All results are dimensionless and can be expressed using four dimensionless parameters (G_d^*, G^*, D^*, and K^*).

S3.1 Model II: considering only dynamic drag force

Displacement of Model II

State 1

$$\begin{aligned} L^* = & \int_0^{t^*} \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)dt^* \\ & +\frac{1}{D^*}\ln\left\{\cos\left[-\sqrt{-G^*D^*}t^*+\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{-\frac{D^*}{G^*}}\right)\right]\right\} \\ & -\frac{1}{D^*}\ln\left[\cos\left(\arctan\left(m-\frac{1}{n}\right)\sqrt{-\frac{D^*}{G^*}}\right)\right] \end{aligned} \tag{S20}$$

State 2

$$\begin{aligned} L^* = & \int_{t_1^*}^{T^*} \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)dt^* \\ & -\frac{1}{D^*}\ln\left\{\cosh\left[-\sqrt{-G^*D^*}T^*+\arctan\left(\frac{1}{n}\sqrt{-\frac{D^*}{G^*}}\right)\right]\right\} \end{aligned} \tag{S21}$$

Relative position

$$\frac{L^*}{S^*}=\frac{\int_0^{T^*}\frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^{\ *}}{2}t^*\right)dt^*}{\int_0^{\frac{2}{G_{\mathrm{d}}^*}}\left(1-\frac{G_{\mathrm{d}}^{\ *}}{2}t^*\right)dt^*}$$
$$-\frac{G_{\mathrm{d}}^{\ *}}{D^*}\ln\left(\cos\left(\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{-\frac{D^*}{G^*}}\right)\right)\right) \tag{S22}$$
$$-\frac{G_{\mathrm{d}}^{\ *}}{D^*}\ln\left(\cosh\left(\sqrt{-G^*D^*}T^*-\arctan\left(m-\frac{1}{n}\right)\left(\sqrt{-\frac{D^*}{G^*}}\right)\right)\right)$$

S3.2 Model III: considering only earth pressure

$K^*>|G^*|$

Displacement of Model III

State 1

$$L^*=\int_0^{t^*}\frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^{\ *}}{2}t^*\right)dt^*$$
$$+\frac{\left(-K^*+G^*\right)}{2}t^{*2}+\left(m-\frac{1}{n}\right)t^* \tag{S23}$$

State 2

$$L^*=\int_{t_1^*}^{t^*}\frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^{\ *}}{2}t^*\right)dt^* \tag{S24}$$

Relative position

$$\frac{L^*}{S^*}=\frac{1}{n}+\left(m-\frac{1}{n}\right)^2\frac{G_{\mathrm{d}}^{\ *}}{K^*-G^*} \tag{S25}$$

$K^*<|G^*|$

Displacement of Model III

State 1

$$L^* = \int_0^{t^*} \frac{1}{n}\left(1 - \frac{G_{\mathrm{d}}{}^*}{2} t^*\right) dt^* + \frac{\left(-K^* + G^*\right)}{2} t^{*2} + \left(m - \frac{1}{n}\right) t^* \tag{S26}$$

State 2

$$L^* = \int_{t_1^*}^{t^*} \frac{1}{n}\left(1 - \frac{G_{\mathrm{d}}{}^*}{2} t^*\right) dt^* + \frac{K^* + G^*}{2}\left(T^{*2} - t_1{}^{*2}\right) - \left(m - \frac{1}{n}\right)\frac{K^* - G^*}{\left(K^* + G^*\right)}\left(T^* - t_1{}^*\right) \tag{S27}$$

Relative position

$$\frac{L^*}{S^*} = \frac{1}{2} m \frac{m - \frac{1}{n}}{K^* - G^*} + \frac{1}{2n}\left(1 - \frac{G_{\mathrm{d}}{}^*}{2}\frac{m - \frac{1}{n}}{K^* - G^*}\right)\left(\frac{\left(m - \frac{1}{n}\right)\frac{K^* + G^*}{K^* - G^*} - \frac{1}{n}}{K^* + G^* - \frac{G_{\mathrm{d}}{}^*}{2n}}\right) \tag{S28}$$

S3.3 Model I: considering both dynamic drag force and earth pressure

$K^*>|G^*|$

Displacement of Model I

State 1

$$L^* = \int_0^{t^*} \frac{1}{n}\left(1 - \frac{G_{\mathrm{d}}{}^*}{2} t^*\right) dt^* + \frac{1}{D^*}\ln\left[\cos\left(-\sqrt{D^*\left(K^* - G^*\right)} t^* + \arctan\left(\left(m - \frac{1}{n}\right)\sqrt{\frac{D^*}{\left(K^* - G^*\right)}}\right)\right)\right] - \frac{1}{D^*}\ln\left[\cos\left(\arctan\left(\left(m - \frac{1}{n}\right)\sqrt{\frac{D^*}{\left(K^* - G^*\right)}}\right)\right)\right] \tag{S29}$$

State 2

$$L^* = \int_{t_1^*}^{t^*} \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)dt^* \tag{S30}$$

Relative position

$$\frac{L^*}{S^*} = \frac{\int_0^{T^*} \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)dt^*}{\int_0^{\frac{2}{G_{\mathrm{d}}^*}}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)dt^*} \tag{S31}$$

$$-\frac{G_d^*}{D^*}\ln\left(\cos\left(\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{K^*-G^*}}\right)\right)\right)$$

$K^*<|G^*|$

Displacement of Model I

State 1

$$\begin{aligned} L^* &= \int_0^{t^*} \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)dt^* \\ &+\frac{1}{D^*}\ln\left[\cos\left(-\sqrt{D^*\left(K^*-G^*\right)}t^*+\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{\left(K^*-G^*\right)}}\right)\right)\right] \\ &-\frac{1}{D^*}\ln\left[\cos\left(\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{\left(K^*-G^*\right)}}\right)\right)\right] \end{aligned} \tag{S32}$$

State 2

$$\begin{aligned} L^* &= \int_{t_1^*}^{t^*} \frac{1}{n}\left(1-\frac{G_{\mathrm{d}}^*}{2}t^*\right)dt^* \\ &-\frac{1}{D^*}\ln\left[\cos\left(\sqrt{-D^*\left(K^*+G^*\right)}t^*-\sqrt{-\frac{K^*+G^*}{K^*-G^*}}\arctan\left(\left(m-\frac{1}{n}\right)\sqrt{\frac{D^*}{\left(K^*-G^*\right)}}\right)\right)\right] \end{aligned} \tag{S33}$$

Relative position

$$\frac{L^*}{S^*} = \frac{\int_0^{T^*} \frac{1}{n}\left(1 - \frac{G_\mathrm{d}^{\;*}}{2} t^*\right) dt^*}{\int_0^{\frac{2}{G_\mathrm{d}^{\;*}}} \left(1 - \frac{G_\mathrm{d}^{\;*}}{2} t^*\right) dt^*}$$

$$- \frac{G_\mathrm{d}^{\;*}}{D^*} \ln\left[\cos\left(\arctan\left(\left(m - \frac{1}{n}\right)\sqrt{\frac{D^*}{\left(K^* - G^*\right)}}\right)\right)\right] \tag{S34}$$

$$- \frac{G_\mathrm{d}^{\;*}}{D^*} \ln\left[\cos\left(\sqrt{-D^*\left(K^* + G^*\right)}t^* - \sqrt{-\frac{K^* + G^*}{K^* - G^*}} \arctan\left(\left(m - \frac{1}{n}\right)\sqrt{\frac{D^*}{\left(K^* - G^*\right)}}\right)\right)\right]$$